

VARIATIONAL ESTIMATES FOR THE BILINEAR ITERATED FOURIER INTEGRAL

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ABSTRACT. We prove pointwise variational L^p bounds for a bilinear Fourier integral operator in a large but not necessarily sharp range of exponents. This result is a joint strengthening of the corresponding bounds for the classical Carleson operator, the bilinear Hilbert transform, the variation norm Carleson operator, and the bi-Carleson operator. Terry Lyons's rough path theory allows for extension of our result to multilinear estimates. We consider our result a proof of concept for a wider array of similar estimates with possible applications to ordinary differential equations.

1. INTRODUCTION

Consider the bilinear iterated Fourier inversion integral

$$(1) \quad B(f_1, f_2)(x) := \int_{\xi_1 < \xi_2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) e^{ix(\xi_1 + \xi_2)} d\xi_1 d\xi_2 .$$

It is a close relative of the bilinear Hilbert transform, and as such satisfies L^p bounds as in [11, 12, 13, 14].

Given any $r \in (0, \infty)$, let T_r denote the following stronger operator

$$(2) \quad \sup_{K, N_0 < \dots < N_K} \left(\sum_{j=1}^K \left| \int_{N_{j-1} < \xi_1 < \xi_2 < N_j} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) e^{ix(\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right|^{r/2} \right)^{2/r} .$$

Thus T_r is a *variation sum* over truncations of B , in particular it dominates both B and the bi-Carleson operator considered in [21], which essentially is the limit case $r \rightarrow \infty$ of T_r .

The main result of our paper is the following theorem:

Theorem 1.1. *Assume that $r > 2$. Then T_r is bounded from $L^{p_1} \times L^{p_2}$ to L^{p_3} provided that $1/p_3 = 1/p_1 + 1/p_2$ and*

$$(3) \quad \max(1, \frac{2r}{3r-4}) < p_1, p_2 \leq \infty \quad , \quad \max(\frac{2}{3}, \frac{r'}{2}) < p_3 < \infty \quad .$$

Besides strengthening [11, 14, 21], Theorem 1.1 also implies a range of the L^p estimates for the variation norm Carleson theorem in [26]. Namely, the variation

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norm Carleson estimate can be obtained by a variant of (2) without the constraint $\xi_1 < \xi_2$, which in turn can be estimated by the sum of (2) and a symmetric version of (2).

The theory of ordinary differential equations with rough driving signals initiated by T. Lyons [17] and developed by many, for example [18] discusses similar expressions as (2) and controls them using additional information on the Fourier transform of f_1, f_2 , allows to bootstrap our main theorem in a certain range of exponents to multi(sub)linear estimates:

Corollary 1.2. *Let $k \geq 3$. For any $r > 0$ let $T_{k,r}$ denote*

$$(4) \quad \sup_{K, N_0 < \dots < N_K} \left(\sum_{j=1}^K \left| \int_{N_{j-1} < \xi_1 < \dots < \xi_k < N_j} \prod_{m=1}^k \widehat{f}_m(\xi_m) e^{ix\xi_m} d\xi_m \right|^{r/k} \right)^{k/r}.$$

Then for every $p \in (\frac{3}{2}, \infty)$ and $r > \max(2, \frac{p}{p-1})$ it holds that

$$(5) \quad \|T_{k,r}[f_1, \dots, f_k]\|_{p/k} \lesssim \prod_{m=1}^k \|f_m\|_p.$$

The end-point version $r = \infty$ of this corollary was posed as a problem in [24].

It should be of interest to study variants of Theorem 1.1 where $e^{ix(\xi_1 + \xi_2)}$ is replaced by $e^{ix(\alpha_1 \xi_1 + \alpha_2 \xi_2)}$ for general real parameters α_1, α_2 and similarly for Corollary 1.2. Such variants of Theorem 1.1 do not hold if $\alpha_1 \alpha_2 (\alpha_1 + \alpha_2) = 0$, the interesting case $\alpha_1 + \alpha_2 = 0$ is discussed in [23]. It is possible that $\alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \neq 0$ is the only constraint towards such variants of Theorem 1.1, this would lead to strong variants of Corollary 1.2 with interesting consequences for rough ordinary differential equations. For our present purpose, we hope that our omission of further parameters α_1, α_2 simplifies the readability of our proof and we defer the discussion of potentially rich ramifications of the theory of general parameters to the future. We refer to [10] for a discussion of a degenerate trilinear variant of (1).

While [26] establishes a sharp range of exponents for the variation norm Carleson operator, we do not prove that the range of exponents of Theorem 1.1 is sharp, though this range is clearly dictated by our method of proof. Part of our ambition was to obtain a sufficiently large range of exponents to allow for instances of Corollary 1.2, for example a point $p_1 = p_2 = 2$ and $2 < r < 3$. By the Hölder inequality, the simpler version of T_r where the bilinear Hilbert symbol $1_{\xi_1 < \xi_2}$ is replaced by 1 is controlled by a product of two variation norm Carleson operators, where the new variation-norm exponents s and t satisfy $2/r = 1/s + 1/t$. In order to apply the known estimates (from [26]) to these variation-norm Carleson operators, we need $p_1 > s'$ and $p_2 > t'$ and which clearly leads to the constraint $p_3 > r'/2$. We also need $s, t > 2$ which together with the relation $2/r = 1/t + 1/s$ leads to $t', s' > \max(1, 2r/(3r-4))$ and from there we then obtain the lower bound constraints for p_1, p_2 .

It is not hard to see that for $r \geq 4$ the range (3) becomes the classical range for the bilinear Hilbert transform (and in particular independent of r). Since variation-norm operators with larger exponents are smaller, we may assume without loss of generality that $2 < r < 4$ in the rest of the paper.

2. OUTLINE OF THE PROOF

By dualization and monotone convergence, we could find measurable functions $K : \mathbb{R} \rightarrow \mathbb{Z}_+$, $N_0(x) \leq N_1(x) \leq \dots$, and $d_0(x), d_1(x), \dots$ such that

- (i) $\sum_{j \geq 0} |d_j|^{r/(r-2)} \equiv 1$;
- (ii) for every x , if $j > K(x)$ then $d_j(x) = 0$ and $N_j(x) = N_{j-1}(x)$; and
- (iii) for every $x \in \mathbb{R}$ we have $|T_r(f_1, f_2)(x)| \lesssim B_r(f_1, f_2)(x)$, where

$$B_r(f_1, f_2)(x) := \sum_{j=1}^K d_j \iint_{N_{j-1} < \xi_1 < \xi_2 < N_j} \prod_{j=1,2} \widehat{f}_j(\xi_j) e^{ix\xi_j} d\xi_1 d\xi_2.$$

Thus, it suffices to prove the desired estimate B_r , provided that the implicit constants depend only on r and p_1, p_2, p_3 . We will fix K , (N_j) , and (d_j) in the rest of the paper. By monotone convergence, we may assume that K , N_j are bounded.

For any $M < N$ we will decompose $1_{M < \xi_1 < \xi_2 < N}$ into three components:

$$m_{CC}(M, N, \xi_1, \xi_2) + m_{BC}(M, N, \xi_1, \xi_2) + m_{LM}(M, N, \xi_1, \xi_2)$$

- m_{BC} captures the singularity along the line segment from (M, M) to (N, N) ,
- m_{CC} captures the singularity along the other two edges of the triangle, and
- m_{LM} is an error term that has two singularities at (M, M) and (N, N) .

Construction of these symbols are detailed in Section 3 using a hybrid of the arguments in [26] and [21]. Applying this decomposition for $(M, N) = (N_{j-1}, N_j)$ and using the triangle inequality, it follows that B_r is controlled by three corresponding bilinear operators. Via standard arguments (detailed in the Appendices), each of these three operators is in turn controlled by a bounded sum of discrete operators, which will be described in Section 4. To prove boundedness of the discrete model operators in the desired L^p ranges, we will use the new L^p theory for outer measures introduced in [6]. It turns out that analogues (for outer measure spaces) of classical singular integral operators arise naturally in our proof, and they are effective tools to handle nested levels of time-frequency analysis. In order to study these operators, we adapted an argument in [6] to prove a Marcinkiewicz interpolation theorem (see Lemma 5.1) for (quasi)sublinear maps between outer measure spaces, which generalizes a simpler interpolation result in [6].

2.1. Notational conventions. By a (standard) dyadic interval we mean $[n2^k, (n+1)2^k)$ for some $n, k \in \mathbb{Z}$. For any $a \in [0, 1)$, by an a -shifted dyadic interval we mean an interval of the form $2^k([n, n+1) + (-1)^k a)$.

For any interval I we denote its midpoint by $c(I)$, its left children by I_- and its right children by I_+ . For any positive C , the C -enlargement of I is defined to be the C -dilation of I from its center and will be denoted by CI . Enlargements of cubes are defined similarly. We will define $\tilde{\chi}_I(x) = (1 + \frac{|x-c(I)|}{|I|})^{-2}$.

For any set $A \subset \mathbb{R}$, we define $D_y A := \{2^y \xi : \xi \in A\}$ the dilation of A relative to the origin. For any number α and any interval I , let $I + \alpha := \{x + \alpha : x \in I\}$.

We say that $A \lesssim B$ if there is an absolute constant $C \in (0, \infty)$ such that $|A| \leq C|B|$. If C depends on t_1, t_2, \dots we will say that $A \lesssim_{t_1, t_2, \dots} B$. (We

sometimes suppress some subscripts if the dependence is not important for the relevant discussion.)

If J is an interval we define $M_J f = \sup_{I \supset J} |I|^{-1} \int_I |f(x)| dx$ for every function f . We will use the following normalizations for inner product and Fourier transforms:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} dx \quad , \quad \widehat{f}(\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx \quad .$$

3. DECOMPOSITION OF THE TRIANGULAR SYMBOL

In this section, we will construct m_{CC} and m_{BC} (m_{LM} is defined using (6)).

3.1. Construction of m_{CC} .

3.1.1. *Decomposition of $1_{M < \xi < N}$.* Below we adapt a decomposition in [26]. We partition (M, N) into maximal dyadic intervals I such that

$$\text{dist}(I, \{M, N\}) \geq L_1 |I| \quad .$$

Let $\mathcal{H} = \mathcal{H}(M, N)$ denote the set of these dyadic intervals. It is clear that the length of two neighboring elements of \mathcal{H} differs by a factor of 2^k where k is integer in $[-1, 1]$, therefore we may denote the lengths of the left and right neighbors of $I \in \mathcal{H}$ by $u(I)|I|$ and $v(I)|I|$ where $u(I), v(I) \in \{\frac{1}{2}, 1, 2\}$ and these two numbers are completely determined from the following details:

- the unique $m \in \mathbb{Z}_+$ such that $M \in I - (m+1)|I|$;
- the unique $n \in \mathbb{Z}_+$ such that $N \in I + (n+1)|I|$;
- whether I is the left or right children of its dyadic parent.

Let A denote the set of eligible $(side, m, n)$ for $I \in \mathcal{H}$, where $side \in \{left, right\}$. For any $\alpha \in A$, let $\mathbf{I}_{M,N}(\alpha)$ be the set of corresponding dyadic intervals.

Given any $1/2 < c < 5/8$ by elementary arguments we could construct smooth functions $\phi_{u,v}$ indexed by $(u, v) \in \{\frac{1}{2}, 1, 2\}^2$, all supported in $[-c, c]$, such that

$$1_{(M,N)}(\xi) = \sum_{I \in \mathcal{H}(M,N)} \phi_{u(I),v(I)}\left(\frac{\xi - c(I)}{|I|}\right) \quad .$$

We will write $\phi_{\alpha,I}(\xi) := \phi_{u,v}(\frac{\xi - c(I)}{|I|})$ if the details of I match with α , thus

$$(6) \quad 1_{(M,N)}(\xi) = \sum_{\alpha \in A} \sum_{I \in \mathbf{I}_{M,N}(\alpha)} \phi_{\alpha,I}(\xi) \quad .$$

Note that the sum in the right converges absolutely pointwise: every term is nonnegative, and at every ξ there are only finitely many nonzero terms in the sum.

3.1.2. *Definition of m_{CC} .* Intuitively, $m_{CC}(M, N, \xi_1, \xi_2)$ is a smooth restriction of $1_{M < \xi_1 < \xi_2 < N}$ to

$$(7) \quad \left\{ (\xi_1, \xi_2) \in [M, N]^2 : \min(|\xi_1 - M|, |\xi_2 - N|) \leq |\xi_1 - \xi_2|/200 \right\} \quad ,$$

which will be denoted by $R_1(M, N)$. For a more precise statement, see Lemma A.4.

To motivate, note that using (6) we obtain for any $(\xi_1, \xi_2) \in R_1$

$$(8) \quad \begin{aligned} 1_{M < \xi_1 < \xi_2 < N} &= 1_{M < \xi_1 < N} 1_{M < \xi_2 < N} \\ &= \sum_{\alpha, \beta \in A} \sum_{I \in \mathbf{I}_{M,N}(\alpha)} \sum_{J \in \mathbf{I}_{M,N}(\beta)} \phi_{\alpha,I}(\xi_1) \phi_{\beta,J}(\xi_2) \end{aligned}$$

and we will construct m_{CC} by removing terms supported far from R_1 in the sum.

Specifically, let

$$A_1 = \{m_\alpha \leq n_\alpha/4\} \quad , \quad A_3 = \{m_\alpha \geq 4n_\alpha\} \quad , \quad A_2 = A - A_1 - A_3 \quad ,$$

which are subsets of A . As we will see, A_2 is a finite set.

Definition 3.1. *Define*

$$m_{CC}(M, N, \xi_1, \xi_2) := \sum_{k=1}^5 m_{CC,k}(M, N, \xi_1, \xi_2)$$

where $m_{CC,j}$ are sub-sums of the right hand side of (8) under some extra constraints. We will always have $I \in \mathbf{I}_{M,N}(\alpha)$ and $J \in \mathbf{I}_{M,N}(\beta)$ and the summands are always $\phi_{\alpha,I}(\xi_1) \phi_{\beta,J}(\xi_2)$, and the following table details the extra constraints:

<i>Symbols</i>	<i>Conditions on α</i>	<i>Conditions on β</i>	<i>Extra conditions on I, J</i>
$m_{CC,1}$	$\alpha \in A_1$	$\beta \in A_1$	$ I \leq J /16$
$m_{CC,2}$	$\alpha \in A_1$	$\beta \in A_2$	$ I \leq J /16$
$m_{CC,3}$	$\alpha \in A_1$	$\beta \in A_3$	<i>None</i>
$m_{CC,4}$	$\alpha \in A_2$	$\beta \in A_3$	$ I \geq 16 J $
$m_{CC,5}$	$\alpha \in A_3$	$\beta \in A_3$	$ I \geq 16 J $

3.2. Construction of m_{BC} . In this section, we construct the m_{BC} symbol, which may be viewed as a smooth restriction of $\chi_{M < \xi_1 < \xi_2 < N}$ to a neighborhood of $R_2(M, N)$, which consists of all $M < \xi_1, \xi_2 < N$ such that

$$(9) \quad |\xi_1 - \xi_2| < \frac{\min(|\xi_1 + \xi_2 - M|, |\xi_1 + \xi_2 - N|)}{100}$$

For a more precise statement, see Lemma A.5.

To motivate the construction, we first note that if $(\xi_1, \xi_2) \in R_2$ then

$$\chi_{M < \xi_1 < \xi_2 < N} = \chi_{\xi_1 < \xi_2} \chi_{M < \xi_1 + \xi_2 < N} \quad .$$

We will construct m_{BC} by writing the product in the right hand side of the last display as a sum of products of wave packets (using a suitable decomposition for each factor), and then removing from this sum essentially those terms that are supported far from R_2 .

3.2.1. *Decomposition of $\chi_{\xi_1 < \xi_2}$.* We largely follow [14] (see also [28, 19]).

Shifted cubes: For any $b = (b_1, \dots, b_n) \in [0, 1]^n$ we say that $S = S_1 \times \dots \times S_n$ is a b -shifted dyadic cube if S_j is a b_j -shifted dyadic interval, and $|S_1| = \dots = |S_n|$. In this paper, unless otherwise specified, the coordinates of underlying shifts are assumed to be in $\{0, \frac{1}{3}, \frac{2}{3}\}$.

Whitney decomposition: For each $b \in \{0, \frac{1}{3}, \frac{2}{3}\}^2$ consider the collection \mathbf{S}_b of all maximal b -shifted squares $S_1 \times S_2$ satisfying

(i) $\text{dist}(S_1 \times S_2, \ell) \geq L_2 |S_1|$, here ℓ is the line $\{\xi_1 = \xi_2\}$.

It is clear that for such square it holds that

$$(10) \quad L_2 |S_1| \leq \text{dist}(S_1 \times S_2, \ell) \leq (2L_2 + 1) |S_1| \quad .$$

Using a partition of unity argument, we may write

$$\chi_{\xi_1 < \xi_2} = \sum_b \sum_{S_1 \times S_2 \in \mathbf{S}_b} \phi_{S_1 \times S_2}(\xi_1, \xi_2)$$

where $\{\phi_{S_1 \times S_2}\}$ is a family of nonnegative C^∞ bump functions, such that $\phi_{S_1 \times S_2}$ is supported inside $\frac{4}{5}S_1 \times \frac{4}{5}S_2$. Note that if $(\xi_1, \xi_2) \in S$ then $\xi_1 + \xi_2 \in S_1 + S_2$. Note that $S_1 + S_2$ can be generously covered by 6 intervals of the form $(3/4)I_j$, $1 \leq j \leq 4$, where I_j are shifted dyadic intervals having the same length as $|S_1|$.

Using partitions of unity, we may find nonnegative smooth functions ϕ_{3, I_j} with support inside $(4/5)I_j$, such that for every $\xi \in S_1 + S_2$ it holds that $1 = \sum_j \phi_{3, I_j}(\xi)$. We obtain

$$\chi_{\xi_1 < \xi_2} = \sum_{S \in \mathbf{S}} \phi_{S_1 \times S_2}(\xi_1, \xi_2) \phi_{3, S_3}(\xi_1 + \xi_2)$$

where \mathbf{S} denote the collection of cubes $S_1 \times S_2 \times S_3$ formed using $S_1 \times S_2 \in \bigcup \mathbf{S}_b$ and S_3 are the covering intervals I_j 's discussed above. Note that every element of \mathbf{S} is a shifted dyadic cube, where the underlying shifts are elements of $\{0, \frac{1}{3}, \frac{2}{3}\}^3$. Expanding $\phi_{S_1 \times S_2}(\xi_1, \xi_2)$ into bilinear Fourier series, we obtain

$$(11) \quad \chi_{\xi_1 < \xi_2}(\xi_1, \xi_2) = \sum_{k \in \mathbb{Z}^2} a_k \sum_{S \in \mathbf{S}} \phi_{1, S, k}(\xi_1) \phi_{2, S, k}(\xi_2) \phi_{3, S, k}(\xi_1 + \xi_2) \quad ,$$

here (a_k) is a rapidly decaying sequence and $\phi_{i, S, k}$'s are C^d -bump functions uniformly adapted to $S \in \mathbf{S}$, and $\phi_{i, S, k}$ is supported inside $\frac{5}{6}S_i$. (In fact $\phi_{3, S, k} \equiv \phi_{3, S_3}$ is independent of k , but we prefer to use $\phi_{3, S, k}$ for later convenience of notation.) Als, d is a fixed finite constant, but could be chosen arbitrarily large.

3.2.2. *Definition of m_{BC} .* For convenience, let $\ell(S) = |S_1| = |S_2| = |S_3|$ denote the side length of S . Using (6) and (11), it follows that we may write $\chi_{\xi_1 < \xi_2}(\xi_1, \xi_2) \chi_{M < \xi_1 + \xi_2 < N}$ as

$$(12) \quad = \sum_{k, \alpha, S, I} a_k \phi_{\alpha, I}(\xi_1 + \xi_2) \phi_{3, S, k}(\xi_1 + \xi_2) \prod_{j=1}^2 \phi_{j, S, k}(\xi_j) \quad .$$

Here the summation is over all $k \in \mathbb{Z}^2$, $\alpha \in A$, $S \in \mathbf{S}$, and $I \in \mathbf{I}_{M, N}(\alpha)$.

Definition 3.2. *Define*

$$m_{BC}(M, N, \xi_1, \xi_2) := \sum_{\substack{k, \alpha, S, I \\ \ell(S) \leq |I|}} a_k \phi_{\alpha, I} \left(\frac{\xi_1 + \xi_2}{2} \right) \phi_{3, S, k}(\xi_1 + \xi_2) \prod_{j=1}^2 \phi_{j, S, k}(\xi_j)$$

4. THE MODEL OPERATORS

4.1. Tiles and wave packets. A (standard) tile $P = I_P \times \omega_P$ is a rectangle of area 1, where the spatial interval I_P is a standard dyadic interval and the frequency interval ω_P is a standard dyadic interval. We will also use shifted tiles, where I_P is still standard dyadic but ω_P is a shifted dyadic interval.

Let \mathbf{P} be a tile collection. For $1 \leq p \leq \infty$ we say that the collection of functions $\{\phi_P, P \in \mathbf{P}\}$ is a L^p -normalized wave packet collection if $\hat{\phi}_P \subset (5/4)\omega_P$, and

$$\frac{d^n}{dx^n} \left(e^{-ic(\omega_P)x} \phi_P(x) \right) \lesssim_n \frac{1}{|I_P|^{n+1/p}} \left(1 + \frac{|x - c(I_P)|}{|I_P|} \right)^{-n}$$

uniformly over $P \in \mathbf{P}$, for all $n \geq 0$. If the estimate holds only for $0 \leq n \leq n_0$ then we say that the collection is of order n_0 .

4.2. Multi-tiles. We say $Q = (Q_1, \dots, Q_m)$ is an m -tile if the tiles Q_1, \dots, Q_m share the same spatial interval, denoted by I_Q . The cube $\omega_Q := \omega_{Q_1} \times \dots \times \omega_{Q_m}$ is called the frequency cube of Q .

Definition 4.1 (Sparse). *A collection \mathbf{Q} of m -tiles is sparse relative to a constant C_0 if the following hold: for any $Q, R \in \mathbf{Q}$ with $|I_R|/C_0 \leq |I_Q| \leq |I_R|$ we must have $|I_Q| = |I_P|$, and furthermore either $\omega_Q = \omega_R$ or $C_0\omega_Q \cap C_0\omega_R = \emptyset$.*

Definition 4.2 (Rank-1). *A collection \mathbf{Q} of m -tiles is of rank 1 relative to $C_1 \geq 1$ if the following holds for any $Q, R \in \mathbf{Q}$:*

- *If there exists j such that $\omega_{Q_j} = \omega_{R_j}$ then $\omega_Q = \omega_R$.*
- *For every $j_0 \in \{1, \dots, m\}$, if $5\omega_{Q_{j_0}} \subset 5\omega_{R_{j_0}}$ then $5C_1\omega_Q \subset 5C_1\omega_R$. If furthermore $|I_R| < |I_Q|$ then $5\omega_{Q_j} \cap 5\omega_{R_j} = \emptyset$ for every $j \neq j_0$.*

4.3. Rigid triples of intervals. We say that $\{(I_{\text{lower}}, I, I_{\text{upper}}), I \in \mathbf{I}\}$ is a rigid collection of interval triples if there are integers $L_1 < m, n \lesssim L_1$ such that one of the following situations happens:

- (i) For every $I \in \mathbf{I}$ we have $I_{\text{lower}} = I - m|I|$ and $I_{\text{upper}} = I + n|I|$.
- (ii) For every $I \in \mathbf{I}$ we have $I_{\text{lower}} = I - m|I|$ and $I_{\text{upper}} = [c(I) + (n-1/2)|I|, \infty)$.
- (iii) For every $I \in \mathbf{I}$ we have $I_{\text{lower}} = (-\infty, c(I) - (m-1/2)|I|)$ and $I_{\text{upper}} = I + n|I|$.

We say that two collections have the same structure if the same situation (i.e. (i) or (ii) or (iii)) holds for both, with possibly different pairs (m, n) .

4.4. Description of the model operators. To bound T_r , we will show that it suffices to bound the following four types of operators: $T_{C \times C}$ (product of Carleson operators), T_{CC} (paraproduct of Carleson operators), T_{BC} (composition of bilinear Hilbert transform and Carleson operators), T_{LM} (variational bi-linear Carleson operators). We will define these operators shortly. For convenience of notation, in the following we denote

$$(13) \quad \tilde{\phi}_{1,P,j}(x) = \phi_{1,P}(x) 1_{N_{j-1}(x) \in \omega_{1,P,lower}} 1_{N_j(x) \in \omega_{1,P,upper}}$$

and we define $\tilde{\phi}_{2,P,j}$, $\tilde{\phi}_{3,P,j}$, $\tilde{\phi}_{1,Q,j}$, ... similarly. As a convention, \mathbf{P} s will denote tile collections and \mathbf{Q} s will denote shifted tri-tile collections, and all underlying collection of wave packets are L^1 -normalized. All interval triples will be rigid, and in type CC we demand that the two underlying rigidity types are the same. Without loss of generality we assume that all tile and tritiles collections are finite and sufficiently sparse (all estimates are uniform over these collections), and for \mathbf{Q} we assume that the tri-tiles share the same shift.

$$\begin{aligned} T_{C \times C}(f_1, f_2) &= \sum_{j=1}^K d_j \left(\sum_{P \in \mathbf{P}_1} |I_P| \langle f_1, \phi_{1,P} \rangle \tilde{\phi}_{1,P,j} \right) \left(\sum_{P \in \mathbf{P}_2} |I_P| \langle f_2, \phi_{2,P} \rangle \tilde{\phi}_{2,P,j} \right) \\ T_{CC}(f_1, f_2) &= \sum_{j=1}^K d_j \sum_{P \in \mathbf{P}_1} |I_P| \langle f_1, \phi_{1,P} \rangle \tilde{\phi}_{1,P,j} \sum_{\substack{P' \in \mathbf{P}_2 \\ |I_{P'}| \leq |I_P|/16}} |I_{P'}| \langle f_2, \phi_{2,P'} \rangle \tilde{\phi}_{2,P',j} \\ T_{BC}(f_1, f_2) &= \sum_{j=1}^K d_j \sum_{Q \in \mathbf{Q}} |I_Q| \langle f_1, \phi_{1,Q} \rangle \langle f_2, \phi_{2,Q} \rangle \sum_{P \in \mathbf{P}: |I_P| \leq |I_Q|} |I_P| \langle \phi_{3,Q}, \phi_P \rangle \tilde{\phi}_{P,j} \\ T_{LM}(f_1, f_2) &= \sum_{j=1}^K d_j \sum_{Q \in \mathbf{Q}} |I_Q| \langle f_1, \phi_{1,Q} \rangle \langle f_2, \phi_{2,Q} \rangle \phi_{3,Q}(x) 1_{N_{j-1}, N_j \text{ constraints}} \cdot \end{aligned}$$

The constraints in T_{LM} read as follow:

- $2N_{j-1} \in \omega_{3,Q,lower}$, $2N_j \in \omega_{3,Q,upper}$,
- $N_{j-1} \in \omega_{k,Q,lower}$ and $N_j \in \omega_{k,Q,upper}$ for each $k = 1, 2$.

Furthermore, in T_{LM} \mathbf{Q} is not shifted and it satisfies the following rigidity constraint: for m_2, m_3 fixed bounded integers it holds for every $Q \in \mathbf{Q}$ that

- ω_{Q_2} is the translation of ω_{Q_1} by $m_2|\omega_{Q_1}|$, and
- $\{x/2 : x \in \omega_{Q_3}\}$ is the translation of the left children of ω_{Q_1} by $m_3|\omega_{Q_1}|/2$,

4.5. Reduction to model operators. In the Appendix, we will show that $B_r(f_1, f_2)$ is bounded by a finite average of discrete operators of the above types. By the Hölder inequality (see also the discussion at the end of the introduction), the desired bounds for $T_{C \times C}$ follow from known L^p estimates for discrete variation-norm Carleson operators [26]: in Section 6 we will also deduce these estimates (see Theorem 6.9) as a byproduct of several generalized Carleson embedding estimates and the L^p theory for outer measure introduced in [6]. The proof of Theorem 6.9 will also serve as a model for the unfortunately more technical treatments for T_{CC} , T_{BC} , and T_{LM} .

5. SOME BACKGROUND ON OUTER MEASURE SPACES

We recall several notions from [6], with some simplifications for the setting of the current paper. An outer measure space (X, S, μ) consists of:

- (i) A countable set X . Often we will assume X is finite, in that case the underlying estimates are independent of the size of X .
- (ii) An outer measure μ generated using countable coverings from a pre-measure on a fixed collection \mathbf{E} of non-empty subsets of X , which in particular covers X .
- (iii) A *size* S which assigns a number in $[0, \infty]$ to each pair (f, E) where $f : X \rightarrow \mathbb{C}$ Borel measurable and $E \in \mathbf{E}$, such that

$$S(f + g)(E) \lesssim S(f)(E) + S(g)(E) \quad , \quad S(\lambda f)(E) = |\lambda| S(f)(E) \quad ,$$

and if $|f| \leq |g|$ then $S(f)(E) \leq S(g)(E)$ for all $E \in \mathbf{E}$.

Given an outer measure space, we may define $\mathcal{L}^\infty \equiv \mathcal{L}^{\infty, \infty}$ by

$$\|f\|_{\mathcal{L}^\infty(X, S, \mu)} = \text{outsup}_X S(f) \equiv \sup_{E \in \mathbf{E}} S(f)(E) \quad ,$$

and \mathcal{L}^p may be defined as follows:

- for Borel measurable $F \subset X$ we define $\text{outsup}_F S(f) := \|f 1_F\|_{\mathcal{L}^\infty(X, S, \mu)}$.
- for any $\lambda \in \mathbb{R}$ let $\mu(S(f) > \lambda) := \inf\{\mu(F) : \text{outsup}_{X \setminus F} S(f) \leq \lambda\}$. Then

$$\begin{aligned} \|f\|_{\mathcal{L}^{p, \infty}(X, S, \mu)} &= \sup_{\lambda > 0} \lambda \mu(S(f) > \lambda)^{1/p} \quad , \\ \|f\|_{\mathcal{L}^p(X, S, \mu)} &= \left(p \int_0^\infty \lambda^{p-1} \mu(S(f) > \lambda) d\lambda \right)^{1/p} . \end{aligned}$$

Many standard properties of classical L^p spaces can be proved for outer L^p spaces, see [6] for details. We summarize several estimates from [6].

Proposition 5.1 (Outer Radon–Nikodym). *Assume that $\|f\|_{\mathcal{L}^\infty(X, S, \mu)} < \infty$, and for some Borel measure ν on X it holds for every $E \in \mathbf{E}$ that*

$$\int_E |f| d\nu \leq C_1 \mu(E) S(f)(E) \quad .$$

Then it holds that (the implicit constant does not depend on $\|f\|_{\mathcal{L}^\infty(X, S, \mu)}$)

$$\int_X |f| d\nu \lesssim_{C_1} \|f\|_{\mathcal{L}^1(X, S, \mu)} \quad .$$

Proposition 5.2 (Outer Hölder). *Suppose that $S(f_1 f_2)(E) \leq \prod_j S_j(f_j)(E)$ for all $E \in \mathbf{E}$. Let $p_1, p_2, p_3 \in (0, \infty]$ such that $1/p_1 + 1/p_2 = 1/p_3$. Then*

$$\|f_1 f_2\|_{\mathcal{L}^{p_3}(X, S, \mu)} \leq 2 \prod_j \|f_j\|_{\mathcal{L}^{p_j}(X, S_j, \mu)} \quad .$$

Proposition 5.3 (Convexity). *If $p_1 < p < p_2$ and $1/p = \alpha_1/p_1 + \alpha_2/p_2$ with $\alpha_1, \alpha_2 \in (0, 1)$ and $\alpha_1 + \alpha_2 = 1$, then*

$$\|f\|_{\mathcal{L}^p(X, S, \mu)} \leq C \prod_j \|f\|_{\mathcal{L}^{p_j, \infty}(X, S_j, \mu)}^{\alpha_j} \quad .$$

The following Lemma generalizes [6, Proposition 3.5]. Below (X, S, μ) and (Y, S', ν) are given outer measure spaces.

Lemma 5.1. *Let K be an operator mapping Borel measurable functions on X to Borel measurable functions on Y , with the following properties:*

- (i) *Scaling invariance: for any $\lambda \geq 0$, $|K(\lambda f)| = |\lambda K(f)|$;*
- (ii) *Quasi sublinear: $|K(f + g)| \leq C|K(f)| + C|K(g)|$;*
- (iii) *Bounded from $L^{p_j}(X, S, \mu)$ to $L^{q_j, \infty}(Y, S', \nu)$, i.e. $\|Kf\|_{\mathcal{L}^{q_j, \infty}} \leq M_j \|f\|_{\mathcal{L}^{p_j}}$.*

Let $p_0 < p_1$ and $q_0 < q_1$ such that $0 < p_j \leq q_j \leq \infty$. For $\theta \in (0, 1)$ assume that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0} \quad , \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_0} \quad .$$

Then K maps $L^p(X, S, \mu)$ into $L^q(Y, S', \nu)$ with norm controlled by $M_0^{1-\theta} M_1^\theta$.

Proof. By scaling invariance, we may assume $M_0 = M_1 = M = 1$. Let

$$\alpha := \left(\frac{q}{p}\right) \frac{1/q_0 - 1/q_1}{1/p_0 - 1/p_1}$$

Case I: $q_1 < \infty$. It follows that $p_1 < \infty$ since $p_j \leq q_j$. Without loss of generality, assume that $\|f\|_{\mathcal{L}^p(X, S, \mu)} = 1$.

For each $\lambda > 0$, by definition there exists a set U such that $S(f1_{U^c}) \leq \lambda^\alpha$ and $\mu(U) \leq 2\mu(S(f) > \lambda^\alpha)$. Let $f_{\lambda, l} = f1_U$ and $f_{\lambda, s} = f1_{U^c}$. Here s stands for small and l stands for large.

Since $f_{\lambda, l}$ is supported on U and $|f_{\lambda, l}| \leq |f|$ and monotonicity of size, we have

$$\begin{aligned} \|f_{\lambda, l}\|_{\mathcal{L}^{p_0}(X, S, \mu)}^{p_0} &\lesssim \left(\int_0^{\lambda^\alpha} + \int_{\lambda^\alpha}^\infty \right) t^{p_0-1} \mu(S(f_{\lambda, l}) > t) dt \\ &\lesssim \lambda^{p_0 \alpha} \mu(S(f) > \lambda^\alpha) + \int_{\lambda^\alpha}^\infty t^{p_0-1} \mu(S(f) > t) dt \quad . \end{aligned}$$

Using $\|f\|_{\mathcal{L}^p} = 1$, it follows in particular that $\|f_{\lambda, l}\|_{\mathcal{L}^{p_0}} \lesssim \lambda^{1-q/q_0}$, and

$$\int_0^\infty \lambda^{(p-p_0)\alpha-1} \|f_{\lambda, l}\|_{\mathcal{L}^{p_0}(X, S, \mu)}^{p_0} d\lambda \lesssim \int_0^\infty \beta^{p-1} \mu(S(f) > \beta) d\beta \lesssim 1 \quad .$$

For $\|f_{\lambda, s}\|_{p_1}$, using $S(f_{\lambda, s}) \leq \lambda^\alpha$ and monotonicity of size we similarly obtain

$$\|f_{\lambda, s}\|_{\mathcal{L}^{p_1}(X, S, \mu)} \lesssim \left(\int_0^{\lambda^\alpha} t^{p_1-1} \mu(S(f) > t) dt \right)^{1/p_1} \lesssim \lambda^{1-q/q_1} \quad ,$$

$$\int_0^\infty \lambda^{(p-p_1)\alpha-1} \|f_{\lambda, s}\|_{\mathcal{L}^{p_1}(X, S, \mu)}^{p_1} d\lambda \lesssim \int_0^\infty \beta^{p-1} \mu(S(f) > \beta) d\beta \lesssim 1 \quad .$$

Since $q_0 \geq p_0$ and $q_1 \geq p_1$, using quasi linearity and scaling invariance and the given assumption on bounds for K at the endpoints, it follows that

$$\begin{aligned} \nu(S'(K(f)) > \lambda) &\lesssim \lambda^{-q_0} \|f_{\lambda, l}\|_{\mathcal{L}^{p_0}(X, S, \mu)}^{q_0} + \lambda^{-q_1} \|f_{\lambda, s}\|_{\mathcal{L}^{p_1}(X, S, \mu)}^{q_1} \\ &\lesssim \lambda^{(p-p_0)\alpha-q} \|f_{\lambda, l}\|_{\mathcal{L}^{p_0}(X, S, \mu)}^{p_0} + \lambda^{(p-p_1)\alpha-q} \|f_{\lambda, s}\|_{\mathcal{L}^{p_1}(X, S, \mu)}^{p_1} \quad , \end{aligned}$$

here we have used the definition of α . It follows that

$$\begin{aligned} \|Kf\|_{\mathcal{L}^q(Y, S', \nu)}^q &\lesssim \int_0^\infty \lambda^{q-1} \nu(S'(K(f)) > \lambda) d\lambda \\ &\lesssim \int_0^\infty \lambda^{(p-p_0)\alpha-1} \|f_{\lambda, l}\|_{\mathcal{L}^{p_0}(X, S, \mu)}^{p_0} d\lambda + \int_0^\infty \lambda^{(p-p_1)\alpha-1} \|f_{\lambda, s}\|_{\mathcal{L}^{p_1}(X, S, \mu)}^{p_1} d\lambda \lesssim 1. \end{aligned}$$

Case I: $q_1 = \infty$. As before, we decompose $f = f_{\lambda, l} + f_{\lambda, s}$ where $S(f_{\lambda, s}) \leq c\lambda^\alpha$ for $c > 0$ small (chosen later), and $f_{\lambda, l}$ is supported in a set U such that $\mu(U) \leq 2\mu(S(f) > c\lambda^\alpha)$. Since $M_1 = 1$, by choosing $c > 0$ sufficiently small we obtain $\nu(S'(Kf) > \lambda) \leq \nu(S'(Kf_{\lambda, l}) \gtrsim \lambda^\alpha)$. The rest of the proof is similar. \square

The following Lemma is a multilinear extension of Lemma 5.1. Below (X, S, μ) and (X_j, S_j, μ_j) are outer measure spaces, $j = 1, \dots, n$. Let $0 < s_j < t_j \leq \infty$, $j = 1, \dots, n$. Let A be the n -dimensional rectangle $\{(x_1, \dots, x_n) : 1/t_j \leq x_j \leq 1/s_j\}$.

Lemma 5.2. *Let K maps measurable function (f_1, \dots, f_n) on $X_1 \times \dots \times X_n$ to measurable functions on X . Assume that K has the following properties:*

(i) *Scaling invariance: for any $\lambda \geq 0$ and $1 \leq j \leq n$,*

$$|K(\dots \lambda f_j \dots)| = |\lambda K(\dots f_j \dots)|$$

(ii) *Quasi sublinear: for any $1 \leq j \leq n$*

$$|K(f_1, \dots, f_j + g_j, \dots, f_n)| \leq C|K(f_1, \dots, f_j, \dots, f_n)| + C|K(f_1, \dots, g_j, \dots, f_n)|$$

(iii) *For every (p, p_1, \dots, p_n) such that $1/p = 1/p_1 + \dots + 1/p_n$ and $(1/p_1, \dots, 1/p_n)$ is one of the vertices of A it holds that*

$$\|Kf\|_{\mathcal{L}^{p, \infty}} \lesssim_{p, p_1, \dots, p_n} \prod \|f_j\|_{\mathcal{L}^{p_j}}$$

Then for every (p, p_1, \dots, p_n) such that $1/p = \sum 1/p_j$ and $(1/p_1, \dots, 1/p_n)$ is in the interior of A it holds that

$$\|Kf\|_{\mathcal{L}^p} \lesssim_{p, p_1, \dots, p_n} \prod \|f_j\|_{\mathcal{L}^{p_j}}.$$

Proof. For simplicity we will show the proof for $n = 2$, the general case is similar.

Let $(1/p_1, 1/p_2)$ be in the interior of A and $f_j \in \mathcal{L}^{p_j}$ for $j = 1, 2$, and assume $1/p = 1/p_1 + 1/p_2$. We normalize $\|f_j\|_{p_j} = 1$ and we will show that $\|K(f_1, f_2)\|_p \lesssim 1$.

Case 1: $t_1, t_2 < \infty$.

For every $\lambda > 0$ consider the decomposition $f_j = f_{j, \lambda, s} + f_{j, \lambda, l}$ where $f_{j, \lambda, l}$ is the restriction of f_j to a some $U_j \subset X_j$ chosen such that $S_j(f_j 1_{U_j^c}) \leq \lambda$ and $\mu_j(U_j) \leq 2\mu_j(S_j(f_j) > \lambda)$. From the proof of Lemma 5.1, using $p_j < t_j$ we have

$$\begin{aligned} \int_0^\infty \lambda^{p_j - t_j - 1} \|f_{j, \lambda, s}\|_{t_j}^{t_j} d\lambda &\lesssim \|f_j\|_{p_j}^{p_j} = 1, \\ \int_0^\infty \lambda^{p_j - t_j - 1} \|f_{j, \lambda, l}\|_{t_j}^{t_j} d\lambda &\lesssim \|f_j\|_{p_j}^{p_j} = 1 \end{aligned}$$

Now, given $\lambda > 0$ we will decompose $K(f_1, f_2)$ by decomposing $f_1 = f_{1, \alpha_1, s} + f_{1, \alpha_1, l}$ and similarly $f_2 = f_{2, \alpha_2, s} + f_{2, \alpha_2, l}$ with $\alpha_1 = \lambda^{p/p_1}$ and $\alpha_2 = \lambda^{p/p_2}$ (clearly $\alpha_1 \alpha_2 = \lambda$). This leads to a decomposition of $K(f_1, f_2)$ into four terms, and we will estimate each of them using the known weaktype estimate at one suitable vertex of A . We

show below the treatment for $K(f_{1,\alpha_1,s}, f_{2,\alpha_2,s})$, for which we will use the vertex (t_1, t_2) . Letting t denote $t_1 t_2 / (t_1 + t_2)$, it follows that

$$\begin{aligned}
\int_0^\infty \lambda^{p-1} \mu(S(K(f_{1,\alpha_1,s}, f_{2,\alpha_2,s}))) > \lambda) d\lambda &\lesssim \int_0^\infty \lambda^{p-t-1} \prod_{j=1,2} \|f_{j,\alpha_j,s}\|_{t_j}^{t_1 t_2 / (t_1 + t_2)} d\lambda \\
&\stackrel{\text{(classical Hölder)}}{\lesssim} \prod_{j=1,2} \left(\int_0^\infty \lambda^{p-t_j-1} \|f_{j,\alpha_j,s}\|_{t_j}^{t_j} d\lambda \right)^{t_{3-j}/(t_1+t_2)} \\
&\stackrel{\text{(using } \lambda^{p-1} d\lambda = C \alpha_j^{p_j-1} d\alpha_j)}{=} C \prod_{j=1,2} \left(\int_0^\infty \alpha_j^{p_j-t_j-1} \|f_{j,\alpha_j,s}\|_{t_j}^{t_j} d\alpha_j \right)^{t_{3-j}/(t_1+t_2)} \\
&\lesssim \prod_{j=1,2} \|f_j\|_{p_j}^{p_j t_{3-j}/(t_1+t_2)} \lesssim 1 .
\end{aligned}$$

The other terms (in the decomposition for $K(f_1, f_2)$) could be treated similarly, thus by quasilinearity of size it follows immediately that

$$\|K(f_1, f_2)\|_{\mathcal{L}^p(X,S,\mu)} = (p \int_0^\infty \lambda^{p-1} \mu(S(K(f_1, f_2))) > \lambda) d\lambda)^{1/p} \lesssim 1 .$$

Case 2: Exactly one of t_1, t_2 is ∞ .

Without loss of generality assume that $t_1 < t_2 = \infty$. In this case we still carry out the same decompositions as before. The two terms that does not involve $f_{2,\alpha_2,s}$ could be treated as before. For $K(f_{1,\alpha_1,s}, f_{2,\alpha_2,s})$ using the assumed weak-type estimate at $(t_1, t_2 = \infty)$ we have

$$\begin{aligned}
\mu(S(K(f_{1,\alpha_1,s}, f_{2,\alpha_2,s}))) > c\lambda &\lesssim \lambda^{-t_1} (\|f_{1,\alpha_1,s}\|_{t_1} \|f_{2,\alpha_2,s}\|_\infty)^{t_1} \\
&\lesssim \alpha_2^{t_1} \lambda^{-t_1} \|f_{1,\alpha_1,s}\|_{t_1}^{t_1} = \alpha_1^{-t_1} \|f_{1,\alpha_1,s}\|_{t_1}^{t_1} ,
\end{aligned}$$

$$\int_0^\infty \lambda^{p-1} \mu(S(K(f_{1,\alpha_1,s}, f_{2,\alpha_2,s}))) > c\lambda) d\lambda \lesssim \int_0^\infty \alpha_1^{p_1-t_1-1} \|f_{1,\alpha_1,s}\|_{t_1}^{t_1} d\alpha_1 \lesssim 1 .$$

The term $K(f_{1,\alpha_1,l}, f_{2,\alpha_2,s})$ could be treated similarly.

Case 2: $t_1 = t_2 = \infty$.

In this case we modify the decompositions slightly so that $S_j(f_{j,\alpha_j,s}) \leq c\alpha_j$ for both $j = 1, 2$, where $c > 0$ is sufficiently small. It follows from the assumed weak-type estimate at $(t_1, t_2) = (\infty, \infty)$ that

$$\|K(f_{1,\alpha_1,s}, f_{2,\alpha_2,s})\|_\infty \lesssim O(c^2 \alpha_1 \alpha_2) = O(c^2 \lambda)$$

therefore by choosing c sufficiently small we obtain, for some $C > 0$ large,

$$\begin{aligned}
\nu(S'(K(f_1, f_2))) > \lambda &\leq \nu(S'(K(f_{1,\alpha_1,s}, f_{2,\alpha_2,s}))) > \lambda/C + \\
&+ \nu(S'(K(f_{1,\alpha_1,l}, f_{2,\alpha_2,s}))) > \lambda/C + \\
&+ \nu(S'(K(f_{1,\alpha_1,s}, f_{2,\alpha_2,l}))) > \lambda/C .
\end{aligned}$$

The three terms on the right hand side could be treated as before. \square

6. GENERALIZED CARLESON EMBEDDINGS AND OUTER L^p ESTIMATES FOR DISCRETE VARIATION-NORM CARLESON OPERATORS

Let $\{\phi_P, P \in X\}$ be L^1 -normalized Fourier wave packets where \mathbf{P} is finite sparse, such that $\text{supp} \hat{\phi}_P \subset (5/4)\omega_P$, and $\{(\omega_{P,lower}, \omega_P, \omega_{P,upper}), P \in \mathbf{P}\}$ is rigid. For simplicity we assume that $\omega_{P,upper}$ is finite and $\omega_{P,lower}$ is a half line (the other settings could be handled similarly). For technical convenience, assume that $1000\omega_P$ is strictly between $\omega_{P,lower}$ and $\omega_{P,upper}$ for every $P \in \mathbf{P}$.

Let f be a Schwarz function on \mathbb{R} , and $\alpha_j : \mathbb{R} \rightarrow [0, \infty]$, and define

$$T_1 f(P) := \langle f, \phi_P \rangle \quad , \quad T_2 f(P) := \left\langle f, \sum_{1 \leq j \leq K} d_j \tilde{\phi}_{P,j} 1_{|I_P| \leq \alpha_j(x)} \right\rangle \quad ,$$

where $\tilde{\phi}_{P,j}$ is defined by (13).

In this section, we consider embedding estimates for T_1 and T_2 from $L^p(\mathbb{R})$ to outer measure spaces on \mathbf{P} . Following [6] we will refer to these estimates as generalized Carleson embeddings.

6.1. Outer measure spaces. For every $P \in \mathbf{P}$, let $\widetilde{\omega}_P$ be the convex hull of $50\omega_P$ and $50\omega_{P,upper}$.

Generating subsets: A nonempty $E \subset \mathbf{P}$ is a generating set (i.e. $E \in \mathbf{E}$) if there exists a dyadic interval I_E and $\xi_E \in \mathbb{R}$ such that for every $P \in E$ we have

$$I_P \subset I_E \quad , \quad \left(\xi_E - \frac{1}{2|I_E|}, \xi_E + \frac{1}{2|I_E|} \right) \subset \widetilde{\omega}_P$$

We say that E is lacunary if furthermore $\xi_E \in 50\omega_{P,upper}$ for every $P \in E$, and E is overlapping if $\xi_E \in \widetilde{\omega}_P \setminus 50\omega_{P,upper}$ for every $P \in E$.

Outer measure: The outer measure μ will be generated from $\mu(E) = \inf \sum_j |I_{E_j}|$, infimum taken over all countable coverings of E by generating sets.

Size: For any $0 < t < \infty$ and any generating set $E \subset \mathbf{P}$, let S_t be the size

$$S_t(f)(E) = \sup_{S \subset E} \left(\frac{1}{|I_S|} \sum_{P \in S} |f(P)|^t |I_P| \right)^{1/t} .$$

If the supremum is taken over only lacunary subsets S , we call $S_{t,lac}$ the corresponding size, and define $S_{t,overlap}$ similarly. When $t = \infty$ the three sizes agree

$$S_{\infty,lac}(f)(E) = S_{\infty,overlap}(f)(E) = S_{\infty}(f)(E) = \sup_{P \in E} |f(P)| .$$

It is clear that S_t and $S_{t,lac}$ and $S_{t,overlap}$ are decreasing functions of t .

Strongly disjointness: Let (E_m) be lacunary. We say they are strongly disjoint if for any $m \neq n$:

- If $P_1 \in E_m$ and $P_2 \in E_n$ such that $10\omega_{P_1} \cap 10\omega_{P_2} \neq \emptyset$ and $|I_{P_1}| \leq |I_{P_2}|$ then $I_{P_1} \cap I_{E_n} = \emptyset$. (In particular $I_{P_1} \cap I_{P_2} = \emptyset$.)

The following estimate follows from a standard argument, see e.g. [26] or [14].

Proposition 6.1. *Let (E_m) be strongly disjoint lacunary and $\mathbf{P}_1 = \bigcup E_m$. Then*

$$\left\| \sum_{P \in \mathbf{P}_1} |I_P| f(P) \phi_P \right\|_{L^2(\mathbb{R})} \lesssim \left(\sum_{P \in \mathbf{P}_1} |I_P| |f(P)|^2 \right)^{1/2} + \sup_{P \in \mathbf{P}_1} |f(P)| \left(\sum_m |I_{E_m}| \right)^{1/2}.$$

6.2. Embeddings for T_1 . The following is a discrete version of [6, Theorem 5.1] and its $L^{2,\infty}$ endpoint is also reformulation of the well-known size lemma in [14].

Theorem 6.1. *For any $2 < p \leq \infty$ it holds that $\|T_1 f\|_{\mathcal{L}^p(\mathbf{P}, S_{2,lac}, \mu)} \lesssim \|f\|_{L^p(\mathbb{R})}$, and the weak-type estimate holds at $p = 2$.*

Proof. Without loss of generality assume $\|f\|_{L^p(\mathbb{R})} = 1$. The endpoint $p = \infty$ is a consequent of Lemma 6.3, so by interpolation it suffices to consider the weak-type estimate at $p = 2$.

Fix any $\lambda > 0$. We will show that there exists $\mathbf{P}' \subset \mathbf{P}$ such that $\mu(\mathbf{P}') \lesssim \lambda^2$ and $\|T_1 f\|_{\mathcal{L}^\infty(\mathbf{P}', S_{2,lac}, \mu)} \leq \lambda$. Without loss of generality assume that $\|T_1 f\|_{\mathcal{L}^\infty} > 2\lambda$. We define $\mathbf{P}' = \bigcup_i F_i$ where F_i are selected as follows:

(i) If there is $E \subset \mathbf{P}$ lacunary with $\left(\frac{1}{|E|} \sum_{P \in E} |T_1(P)|^2 |I_P| \right)^{1/2} > \lambda$, we select one such E_1 with smallest possible ξ_E .

(ii) Let $F_1 = \{P \in \mathbf{P} : (\xi_{E_1} - \frac{1}{2|I_{E_1}|}, \xi_{E_1} + \frac{1}{2|I_{E_1}|}) \in \tilde{\omega}_P, I_P \subset I_{E_1}\}$.

We remove F_1 from \mathbf{P} and repeat the above argument and select $E_m, F_m, m = 2, 3, \dots$. Since \mathbf{P} is finite the process will stop, and by geometry and sparseness of \mathbf{P} it is clear that $(E_m)_{i \geq 1}$ is strongly disjoint. Let $\mathbf{P}_1 = \bigcup E_m$. It follows that

$$\begin{aligned} \lambda^2 \sum_m |I_{E_m}| &\lesssim \sum_{P \in \mathbf{P}_1} |I_P| |\langle f, \phi_P \rangle|^2 \\ &\lesssim \left\| \sum_{P \in \mathbf{P}_1} |I_P| \langle f, \phi_P \rangle \phi_P \right\|_2 \quad (\text{Cauchy-Schwarz, } \|f\|_2 = 1) \\ &\lesssim \left(\sum_m |I_{E_m}| \right)^{1/2} \|T_1 f\|_{\mathcal{L}^\infty(\mathbf{P}, S_{2,lac}, \mu)} \quad (\text{using Lemma 6.1}) \\ &\lesssim \lambda \left(\sum_m |I_{E_m}| \right)^{1/2} \quad (\text{since } \|T_1 f\|_{\mathcal{L}^\infty} < 2\lambda). \end{aligned}$$

It follows that $\lambda^2 \mu(\mathbf{P}') \leq \lambda^2 \sum_m |I_{E_m}| \lesssim 1$, as desired. \square

For any g consider the following tile maximal average

$$(14) \quad M_N(\mathbf{P}, f) := \sup_{P \in \mathbf{P}} \frac{1}{|I_P|} \int \tilde{\chi}_{I_P}^N f.$$

The following Lemmas are reformulation of standard estimates, see e.g. [20].

Lemma 6.2. *It holds that*

$$\|T_1 f\|_{\mathcal{L}^\infty(\mathbf{P}, S_{2,lac}, \mu)} \lesssim \sup_{E \subset \mathbf{P} \text{ lacunary}} \frac{1}{|I_E|} \left\| \left(\sum_{P \in E} |T_1 f(P)|^2 1_{I_P} \right)^{1/2} \right\|_{L^{1,\infty}}.$$

By Calderon–Zygmund theory, for every lacunary generating set E it holds that

$$\left\| \left(\sum_{P \in E} |T_1 f(P)|^2 1_{I_P} \right)^{1/2} \right\|_{L^{1,\infty}} \lesssim_N \|\tilde{\chi}_{I_E}^N f\|_1$$

Consequently, Lemma 6.3 implies the following standard corollary:

Lemma 6.3. *It holds that*

$$\|T_1 f\|_{\mathcal{L}^\infty(\mathbf{P}, S_{2, lac}, \mu)} \lesssim M_{N+4}(\mathbf{P}, f) \lesssim \sup_{P \in \mathbf{P}} \inf_{x \in I_P} M_{I_P}(\tilde{\chi}_{I_P}^N f)(x) .$$

When \mathbf{P} is a generating set, we have the following standard estimates, part (i) is a reformulation of [25, Proposition 3.4]. For convenience, we will sketch a proof.

Lemma 6.4. (i) *Suppose that E is a generating set. Then for every $1 < p \leq \infty$*

$$\|T_1 f\|_{\mathcal{L}^p(E, S_{2, lac}, \mu)} \lesssim_{p, N} \|f \tilde{\chi}_{I_E}^N\|_p ,$$

and the weak-type endpoint $p = 1$ holds.

(ii) *If E is lacunary then for every $1 < p < \infty$*

$$\left\| \sum_{P \in E} |I_P| T_1 f(P) \phi_P \right\|_{L^p(\mathbb{R})} \lesssim_p \|T_1 f\|_{\mathcal{L}^p(E, S_{2, lac}, \mu)} .$$

Proof. (i) For any $F \subset E$ by Lemma 6.3 and Lemma 6.2 we have:

$$\|T_1 f\|_{\mathcal{L}^\infty(F, S_{2, lac}, \mu)} \leq C_N \sup_{S \subset F \text{ lacunary}} \inf_{x \in I_S} M(f \tilde{\chi}_{I_E}^N)(x) ,$$

thus the endpoint $p = \infty$ follows. By interpolation it suffices to consider the weak-type endpoint at $p = 1$. Fix $\lambda > 0$. We select a sequence of generating subsets S_1, S_2, \dots of E as follows. If there exists $J \subset I_E$ dyadic such that $\inf_{x \in J} M(f \tilde{\chi}_{I_E}^N)(x) > \lambda/C_N$ we choose a maximal J , and let $S_1 = \{P \in E : I_P \subset J\}$ and remove S_1 from E , then repeat the above selection algorithm. Since E is finite the algorithm will stop and we obtain our sequence S_1, S_2, \dots, S_k . Clearly on $E - S_1 - \dots - S_k$ we have $S_{2, lac}(T_1 f) \leq \lambda$. Now, I_{S_j} 's are pairwise disjoint and S_j are generating sets, therefore

$$\lambda \mu\left(\bigcup_j S_j\right) \lesssim \lambda |\{M(f \tilde{\chi}_{I_E}^N) > C_N \lambda\}| \lesssim \|f \tilde{\chi}_{I_E}^N\|_1 .$$

(ii) Let $g \in L^{p'}$, by outer Radon-Nikodym/Hölder we have

$$\begin{aligned} \left\langle \sum_{P \in E} |I_P| T_1 f(P) \phi_P, g \right\rangle &\lesssim \|T_1 f\|_{\mathcal{L}^p(E, S_{2, lac}, \mu)} \|T_1 g\|_{\mathcal{L}^{p'}(E, S_{2, lac}, \mu)} \\ &\lesssim_p \|T_1 f\|_{\mathcal{L}^p(E, S_{2, lac}, \mu)} \|g\|_{L^{p'}(\mathbb{R})} \end{aligned}$$

thanks to part (i). The desired estimate follows from duality. \square

6.3. Embeddings for T_2 . The following Theorem generalizes the variation-norm density lemma in [26], which in turn generalizes the density lemma in [14].

Theorem 6.5. *Assume that $s \neq 2$, and let $q = \min(2, s')$, $s' = s/(s-1)$.*

For any $p \in (s, \infty]$ it holds that

$$\|T_2 f\|_{\mathcal{L}^p(\mathbf{P}, S_{1, overlap} + S_{q', lac}, \mu)} \lesssim M_N(\mathbf{P}, 1_{\text{supp}(f)})^{1/p'} \|f(\sum_j |d_j|^s)^{1/s}\|_p ,$$

and the weak-type estimate holds at $p = s$.

By interpolation, it suffices to consider the L^∞ estimate and the weaktype estimate at $p = s$. (Note that we could fix $G = \text{supp}(f)$ and invoke interpolation theorems for the linear map T_2 from functions on G to outer measure spaces, the factor $M_N(\mathbf{P}, 1_G)^{1/p'}$ should be thought of as an estimate for the norm of T_2 .)

The proof strategy will involve three Lemmas: Lemma 6.7 and Lemma 6.8 will be used to handle the L^∞ endpoint, and Lemma 6.6 will be used to handle the weak type estimate at $p = 2$.

For convenience of notation, for each \mathbf{P} let $\bar{\mathbf{P}}$ denote the following completion

$$(15) \quad \bar{\mathbf{P}} = \{R : \exists P_1, P_2 \in \mathbf{P} \text{ with } I_{P_1} \subset I_R \subset 30I_{P_2}, \tilde{\omega}_R \cap \tilde{\omega}_{P_j} \neq \emptyset\}$$

For convenience of notation, for any measurable sequence (g_j) we denote

$$(16) \quad m_{s,N}(\mathbf{P}, (g_j)) := \sup_{R \in \bar{\mathbf{P}}} \frac{1}{|I_R|} \int \tilde{\chi}_{I_R}^N \left(\sum_{j: N_j \in \tilde{\omega}_R} |g_j|^s \right)^{1/s},$$

Clearly, $m_{\infty,N} \lesssim m_{s,N}$.

Now, the L^∞ endpoint of Theorem 6.5 follows from Using Lemma 6.7 and Lemma 6.8, which gives the estimate

$$\|T_2 f\|_{\mathcal{L}^\infty(\mathbf{P}, S_{1, \text{overlap}} + S_{q', \text{lac}}, \mu)} \lesssim m_{s,N}(\mathbf{P}, (fd_j)) .$$

The weaktype estimate at $p = s$ of Theorem 6.5 follows from the following result.

Lemma 6.6. *Let $p \in [s, \infty]$ and assume $\text{supp}(\sum_j |g_j|^s)^{1/s} \subset G$. For any $\lambda > 0$ there exists $\mathbf{P}_1 \subset \mathbf{P}$ such that*

$$\lambda \mu(\mathbf{P} \setminus \mathbf{P}_1)^{1/p} \lesssim_{N,p,s} M_N(\mathbf{P}, 1_G)^{1/p'} \left\| \left(\sum_j |g_j|^s \right)^{1/s} \right\|_{L^p(\mathbb{R})}$$

$$\text{and} \quad m_{s,N}(\mathbf{P}, (g_j)) \leq \lambda .$$

Proof. The endpoint $p = \infty$ is trivial, while the endpoint $p = s$ follows from

$$m_{s,N}(\mathbf{P}, (g_j)) \lesssim M_N(\mathbf{P}, 1_G)^{1/s'} \sup_{R \in \bar{\mathbf{P}}} \left(\frac{1}{|I_R|} \int \tilde{\chi}_{I_R}^N \sum_{j: N_j \in \tilde{\omega}_R} |g_j|^s \right)^{1/s}$$

and a variation-norm version of the standard density lemma (see e.g. [26]). The general case could be obtained by a simple interpolation argument: let $A = \{x : (\sum_j |g_j(x)|^s)^{1/s} > \lambda/C\}$ for $C > 0$ large, we use the $p = s$ and $p = \infty$ endpoints to respectively treat $g_j 1_A$ and $g_j 1_{A^c}$. We omit the details. \square

Lemma 6.7. *Assume that $s \neq 2$. Let $q = \min(2, s')$. Then*

$$\|T_2 f\|_{\mathcal{L}^\infty(\mathbf{P}, S_{q', \text{lac}}, \mu)} \lesssim m_{s,N}(\mathbf{P}, (fd_j)) .$$

Proof. It suffices to show that for every lacunary E and $g : E \rightarrow \mathbb{C}$ it holds that

$$(17) \quad \sum_{P \in E} |I_P| T_2 f(P) g(P) \lesssim |I_E| \|g\|_{\mathcal{L}^\infty(E, S_{q, \text{lac}}, \mu)} m_{s,N}(E, (fd_j)) .$$

Indeed, taking $g(P) = \overline{T_2 f(P)} |T_2 f(P)|^{q'-2}$ and applying the above estimate for all lacunary subsets of \mathbf{P} , we obtain an equivalent form of the desired estimate:

$$\left(S_{q', \text{lac}}(T_2 f)(\mathbf{P}) \right)^{q'} \lesssim \left(S_{q', \text{lac}}(T_2 f)(\mathbf{P}) \right)^{q'-1} m_{s,N}(\mathbf{P}, (fd_j)) .$$

Below, for brevity we write $\|g\|_\infty$ for $\|g\|_{\mathcal{L}^\infty(E, S_{q, lac}, \mu)}$, and m_s for $m_{s, N}$.

For every P and x , it is clear that at most one j will satisfy $N_{j-1} \in \omega_{P, lower}$ and $N_j \in \omega_{P, upper}$. Let $d_P(x) = d_j(x)1_{|I_P| \leq \alpha_j}$ if such j exists, otherwise $d_P(x) = 0$. Let \mathbf{J} be the collection of all maximal dyadic J such that $3J$ does not contain any I_P , $P \in E$. Clearly, \mathbf{J} is a partition of \mathbb{R} , so the left hand side of (17) is bounded by

$$(18) \quad \leq \sum_{J \in \mathbf{J}} \sum_{P \in E} |I_P| g(P) \phi_P f d_P \|_{L^1(J)} \leq A + B,$$

where A denotes the contribution of (J, P) such that $|I_P| \leq 2|J|$ and the rest is in B . Since $\|\tilde{\chi}_{I_P}^N f d_P\|_1 \lesssim |I_P| m_{\infty, N}(E, (f d_j))$, we obtain

$$\begin{aligned} A &\lesssim \sup_P |g(P)| m_\infty(E, (f d_j)) \sum_{J \in \mathbf{J}} \sum_{\substack{P: |I_P| \leq 2|J| \\ I_P \not\subset 3J}} |I_P| \sup_{x \in J} \tilde{\chi}_{I_E}(x)^2 \tilde{\chi}_{I_P}(x)^2 \\ &\lesssim \sup_P |g(P)| m_\infty(E, (f d_j)) \sum_{J \in \mathbf{J}} \sup_{x \in J} \tilde{\chi}_{I_E}(x)^2 \\ &\lesssim |I_E| \sup_P |g(P)| m_\infty(E, (f d_j)). \end{aligned}$$

Note that the above estimate remains true when E is overlapping.

We now estimate B i.e. the contribution of (J, P) with $|I_P| \geq 4|J|$.

First, by geometry, such J must be a subset of $3I_E$. Furthermore, by maximality there is $P_1 \in E$ such that $I_{P_1} \subset 3\pi(J)$ where $\pi(J)$ is the dyadic parent of J . Let R be a tile such that $(I_{P_1} \cup \pi(J)) \subset I_R$ and $|I_R| = 4|J|$, and $\xi_E \in \omega_R$. We will show that $R \in \bar{\mathbf{P}}$. To see this, note that for (17) we may assume that $I_{P_2} = I_E$ for some $P_2 \in E$. It follows that $I_{P_1} \subset I_R \subset 10J \subset 30I_{P_2}$; it is also clear that $\tilde{\omega}_R \cap \tilde{\omega}_{P_j} \neq \emptyset$ for each $j = 1, 2$ since they all contain ξ_E , thus $R \in \bar{\mathbf{P}}$ as claimed.

Now, if $P \in E$ such that $|I_P| \geq |I_R| = 4|J|$, by sparseness it follows that $\omega_{P, upper} \subset \tilde{\omega}_R$. Now, for $x \in J$ by Hölder's inequality we have

$$\begin{aligned} &\sum_{P \in E: |I_P| > 2|J|} |I_P| g(P) \phi_P(x) f(x) d_P(x) \\ &\lesssim \left(\sum_{j: N_j \in \tilde{\omega}_R} |f d_j|^s \right)^{1/s} \left(\sum_j \sum_{\alpha_j(x) \geq |I_P| > 2|J|} |I_P| g(P) \tilde{\phi}_{P, j}^{s'} \right)^{1/s'}. \end{aligned}$$

Since T is lacunary and \mathbf{P} is sparse, we can find a sequence of integers $O(1) + \log_2(|J|) \leq m_1(x) \leq n_1(x) \leq \dots \leq m_K(x) \leq n_K(x)$ such that

$$\{\alpha_j \geq |I_P| > 2|J| : N_{j-1} \in \omega_{P, lower}, N_j \in \omega_{P, upper}\} = \{P \in E : 2^{m_j} < |I_P| \leq 2^{n_j}\}.$$

(Note that when $m_j = n_j$ the set is understood to be empty, one example when this may happen is when α_j is too small or $2|J|$ is too large relative to the range of $|I_P|$ imposed by the N_j, N_{j-1} constraints.) Now, let $g_E(x) = \sum_{P \in E} |I_P| g(P) \phi_P(x)$. For every $x \in J$ we have

$$\left(\sum_{1 \leq j \leq K} \sum_{\alpha_j \geq |I_P| > 2|J|} |I_P| g(P) \tilde{\phi}_{P, j}^{s'} \right)^{1/s'} = \left(\sum_{1 \leq j \leq K} |(\Pi_{n_j} - \Pi_{m_j}) g_E|^{s'} \right)^{1/s'}$$

where Π_n s are Fourier projections onto the relevant frequency scales of E (essentially projecting onto $|\xi - \xi_E| \lesssim 2^{-n}$, thus larger values of n means narrower

bands). Using $m_1 > \log_2 |J| + O(1)$ and Minkowski's inequality, the last display is bounded by $M_J(\|\Pi_k g_E\|_{V_k^q(\mathbb{Z})})$. It follows that

$$\begin{aligned} B &\lesssim \sum_{J \in \mathbf{J}: J \subset 3I_E} M_J(\|\Pi_k g_E\|_{V_k^{s'}(\mathbb{Z})}) \left(\sum_{j: N_j \in \tilde{\omega}_R} |f d_j|^s \right)^{1/s} \|_{L^1(J)} \\ &\lesssim \|M(\|\Pi_k g_E\|_{V_k^{s'}(\mathbb{Z})})\|_{L^1(3I_E)} m_s(E, (f d_j)) . \end{aligned}$$

Since $s \neq 2$ we either have $s' > 2$ or $s' < 2$. If $s' > 2$ then by the (continuous) Lépingle inequality [15, 1, 9] we obtain

$$\begin{aligned} B &\lesssim |I_E|^{1/2} \|g_E\|_2 m_s(E, (f d_j)) \\ &\lesssim |I_E| \|g\|_{L^\infty(S_{2, lac})} m_s(E, (f d_j)) . \end{aligned}$$

If $s' < 2$ then by the continuous Pisier–Xu inequality (see [5]) we obtain

$$\begin{aligned} B &\lesssim |I_E|^{1/s} \left(\sum_k \left| \sum_{P \in E: |I_P|=2^k} |I_P| g(P) \phi_P \right|^{s'} \right)^{1/s'} m_s(E, (f d_j)) \\ &\lesssim |I_E| \|g\|_{\mathcal{L}^\infty(E, S_{s', lac}, \mu)} m_s(E, (f d_j)) . \quad \square \end{aligned}$$

Lemma 6.8. *If $1 < q \leq \infty$ then*

$$\|T_2 f\|_{\mathcal{L}^\infty(\mathbf{P}, S_{1, overlap}, \mu)} \lesssim m_{\infty, N}(\mathbf{P}, (f d_j)) .$$

Proof. The proof is entirely similar to the proof of Lemma 6.7, it suffices to show for any overlapping E the following estimates

$$(19) \quad \sum_{P \in E} |I_P| T_2 f(P) g(P) \lesssim |I_E| \sup_P |g(P)| m_{\infty, N}(E, (f d_j)) .$$

We define \mathbf{J} as before and invoke (18) again, and A could be estimated as before. To estimate B we use the following observation: since E is overlapping, for every x there is at most one j and at most one scale of E that contributes to $\sum_{|I_P| > 2|J|} |I_P| g(P) \phi_P(x) d_P(x)$. Let $R \in \bar{E}$ be as before. It follows that

$$\begin{aligned} B &\lesssim \sum_{J \subset 3I_E} \left\| \sup_{j: N_j \in \tilde{\omega}_R} \sup_{\alpha > 2|J|} \sum_{P \in E: |I_P|=\alpha} |g(P)| \tilde{\chi}_{I_P}^{N+4} d_j f \right\|_{L^1(J)} \\ &\lesssim \sum_{J \subset 3I_E} \left\| \sup_{j: N_j \in \tilde{\omega}_R} |d_j f| \right\|_{L^1(J)} \sup_P |g(P)| \\ &\lesssim_N |I_E| \sup_P |g(P)| m_{\infty, N}(E, (f d_j)) . \quad \square \end{aligned}$$

6.4. Outer L^p estimates for discrete variation-norm Carleson operators.

Let $r \neq 2$ and $q = \min(2, r)$, and $\tilde{\phi}_{P,j}$'s are defined relative to $(N_j(x))$ and $K(x)$. We consider the variation-norm operator

$$V_{s'}(g) = \left(\sum_j \sup_{\alpha} \left| \sum_{P \in \mathbf{P}: |I_P| \leq \alpha} |I_P| g(P) \tilde{\phi}_{P,j} \right|^r \right)^{1/r}$$

Theorem 6.9. *Let F be a subset of \mathbb{R} .*

(i) *For any $1 < p < r$ it holds that*

$$\|V_r(g)\|_{L^p(F)} \lesssim_{p, N} M_N(\mathbf{P}, 1_F)^{1/p} \|g\|_{\mathcal{L}^p(\mathbf{P}, S_{q, lac}, \mu)} .$$

(ii) If $r > 2$ then for all $p > r'$ it holds that

$$\|V_r(T_1 f)\|_p \lesssim \|f\|_{L^p(\mathbb{R})}.$$

Proof. (i) Let $s = r'$. We may find (α_j) and (g_j) measurable functions with $\alpha_j \geq 0$ and $\sum_{1 \leq j \leq K} |g_j|^s = O(1)$ such that

$$V_r(g) \lesssim \sum_P |I_P| g(P) \sum_j \tilde{\phi}_{P,j} 1_{|I_P| \leq \alpha_j(x)} g_j$$

Let $h \in L^{p'}(\mathbb{R})$ and $T_2 h(P) = \left\langle \sum_j \tilde{\phi}_{P,j} g_j 1_{|I_P| \leq \alpha_j(x)}, h 1_F \right\rangle$. It suffices to show that

$$\sum_P |I_P| g(P) T_2 h(P) \lesssim_{p,N} M_N(\mathbf{P}, 1_F)^{1/p} \|g\|_{\mathcal{L}^p(\mathbf{P}, S_{q,lac}, \mu)} \|h\|_{L^{p'}(\mathbb{R})}$$

Via applications of the classical Hölder inequality it follows that

$$S_1(g T_2 h) \lesssim S_{2,lac}(g)(S_{2,lac} + S_{1,overlap})(T_2 h).$$

Thus, using outer Radon-Nykodym and outer Hölder inequalities, we obtain

$$(20) \quad \sum_P |I_P| g(P) T_2 h(P) \lesssim \|g\|_{\mathcal{L}^p(\mathbf{P}, S_{q,lac}, \mu)} \|T_2 h\|_{\mathcal{L}^{p'}(\mathbf{P}, S_{q',lac} + S_{1,overlap}, \mu)}.$$

Now, using Theorem 6.5 and noticing $s' = r$ it follows that

$$\begin{aligned} \|T_2 h\|_{\mathcal{L}^{p'}(\mathbf{P}, S_{q',lac} + S_{1,overlap}, \mu)} &\lesssim M_N(\mathbf{P}, 1_F)^{1/p} \|h(\sum_j |u_j|^s)^{1/s}\|_{L^{p'}(\mathbb{R})} \\ &\lesssim M_N(\mathbf{P}, 1_F)^{1/p} \|h\|_{L^{p'}(\mathbb{R})}, \end{aligned}$$

provided that $p' > s$. This completes the proof of part (i).

(ii) Since $r > 2$, we obtain $q = 2$. We say that a subset is major if it has at least half of the total measure. By restricted weak type interpolation [22] (see also Section 7.3.2), it suffices to show that we could find β arbitrarily close to 0 and also arbitrarily close to $1/s$ such that $\langle V_r(T_1 f), h \rangle$ is of restricted weak type $(\beta, 1 - \beta)$. That is, there exists $j_0 \in \{1, 2\}$ depending only on β such that given any $F_1, F_2 \subset \mathbb{R}$ with finite positive Lebesgue measures we could find $S_1 \subset F_1$ and $S_2 \subset F_2$ both major subsets and furthermore $S_{j_0} = F_{j_0}$ and

$$\langle V_r(T_1 f), h \rangle \lesssim |F_1|^\beta |F_2|^{1-\beta} \quad \text{if } |f| \leq 1_{S_1} \text{ and } |h| \leq 1_{S_2}.$$

To get β near $1/s$, we let $S_1 = F_1$ and $S_2 = F_2 \setminus E$ where $E := \{M 1_{F_1} > C |F_1|/|F_2|\}$ and C is large enough, and $S_2 = F_2$. Without loss of generality assume that $1 + \frac{\text{dist}(I_P, E^c)}{|I_P|} \sim 2^k$ provided that we have enough decay in the estimate. By convexity, for $2 < p < s'$ it follows from part (i) that

$$\begin{aligned} \langle V_{s'}(g), h \rangle &\lesssim \|g\|_{\mathcal{L}^p(\mathbf{P}, S_{2,lac}, \mu)} M_N(\mathbf{P}, 1_{F_2})^{1/p} \|h\|_{p'} \\ &\lesssim \|g\|_{\mathcal{L}^\infty(\mathbf{P}, S_{2,lac}, \mu)}^{1-\frac{2}{p}} \|g\|_{\mathcal{L}^{2,\infty}(\mathbf{P}, S_{2,lac}, \mu)}^{\frac{2}{p}} M_N(\mathbf{P}, 1_{F_2})^{1/p} |F_2|^{1/p'} \\ &\lesssim 2^{-Nk} (\sup_{P \in \mathbf{P}} \sup_{x \in I_P} M 1_{F_1}(x))^{1-\frac{2}{p}} |F_1|^{1/p} |F_2|^{1/p'} \\ &\lesssim 2^{-k} (|F_1|/|F_2|)^{1-2/p} |F_1|^{1/p} |F_2|^{1/p'} = 2^{-k} |F_1|^\beta |F_2|^{1-\beta}, \end{aligned}$$

where $\beta := 1/p'$, which is arbitrarily close to $1/s$ if p is sufficiently close to s' .

To get β near 0, we let $S_1 = F_1 \setminus E$ and $S_2 = F_2$ where $E := \{M1_{F_2} > C|F_2|/|F_1|\}$ with C sufficiently large. Similarly we obtain

$$\begin{aligned} \langle V_{s'}(g), h \rangle &\lesssim \|g\|_{\mathcal{L}^\infty(\mathbf{P}, S_{2, lac}, \mu)}^{1-\frac{2}{p}} \|g\|_{\mathcal{L}^{2, \infty}(\mathbf{P}, S_{2, lac}, \mu)}^{\frac{2}{p}} M_N(\mathbf{P}, 1_{F_2})^{1/p} |F_2|^{1/p'} \\ &\lesssim 2^{-Nk} |F_1|^{1/p} (2^k |F_2|/|F_1|)^{1/p} |F_2|^{1/p'} \\ &\lesssim 2^{-k} |F_2| \quad . \quad \square \end{aligned}$$

7. ESTIMATES FOR BC MODEL OPERATORS

Theorem 7.1. *Let T be a BC model operator. Then $\|T\|_{L^{q_1} \times L^{q_2} \rightarrow L^q} < \infty$ for every $1/q = 1/q_1 + 1/q_2$ such that $\frac{2r}{3r-4} < q_1, q_2 \leq \infty$ and $q > \frac{r'}{2}$.*

For simplicity, we assume that $\omega_{P, upper}$ is finite and $\omega_{P, lower}$ is a halflife for each $P \in \mathbf{P}$; other situations are either symmetric or could be reduced to this setting.

The outer measure spaces on \mathbf{P} are defined as in Section 6.1. Below we discuss the outer measure spaces on \mathbf{Q} , which are similar to the settings on \mathbf{P} , thus we only discuss the needed changes. By further decomposition if necessary, we may assume that \mathbf{P} and \mathbf{Q} are very sparse.

For any two tiles R, R' we say that $R < R'$ if $I_R \subsetneq I_{R'}$ and $5\omega_{R'} \subset 5\omega_R$.

We say that $R \leq R'$ if $R < R'$ or $R = R'$. Clearly, \leq and $<$ are transitive.

Generating subsets of \mathbf{Q} : A nonempty $E \subset \mathbf{Q}$ is a generating set if for some tritile Q_E (with the same rigidity) the following holds: for every $Q \in E$ there is $j = j(Q) \in \{1, 2, 3\}$ such that $Q_j \leq Q_{E,j}$, where Q_j and $Q_{E,j}$ are the j -tiles of Q and Q_E . We denote $I_E \equiv I_{Q_E}$ and $\xi_E = c(\omega_{Q_E})$.

For any fixed $j \in \{1, 2, 3\}$, we say that E is j -overlapping if $j(Q) = j$ for every $Q \in E$. We say that E is j -lacunary if it is k overlapping for some $k \in \{1, 2, 3\} \setminus \{j\}$.

Outer measure on \mathbf{Q} : Let σ be generated from $\sigma(E) = \inf \sum_j |I_{E_j}|$, infimum taken over all countable coverings of E by generating sets.

Size: For every $0 < t \leq \infty$, we define S_t just as in the setting for \mathbf{P} . If the defining supremum is taken over all j -lacunary S , we obtain $S_t^{[j]}$. Similarly, for $S_t^{[j_1, j_2]}$ the supremum is taken over all S that is both j_1 and j_2 lacunary, or equivalently k overlapping where $k \neq j_1, j_2$.

Strongly disjointness: For any $j \in \{1, 2, 3\}$, a collection (E_m) of j -lacunary sets of tritiles is strongly disjoint if the following holds for any $E_m, E_n, m \neq n$:

- If $Q \in E_m$ and $R \in E_n$ such that $3\omega_{Q_j} \cap 3\omega_{R_j} \neq \emptyset$ and $|I_Q| \leq |I_R|$ then $I_Q \cap I_{E_n} = \emptyset$. (In particular $I_Q \cap I_R = \emptyset$.)

An analogue of Lemma 6.1 also holds in the current setting.

In the rest of this section, for every $E \subset \mathbf{Q}$ let \mathbf{P}_E denote the set of all $P \in \mathbf{P}$ such that there exists at least one $Q \in E$ with $|I_P| \leq |I_Q|$ and $\frac{5}{4}\omega_P \cap \frac{5}{6}\omega_{3,Q} \neq \emptyset$. The following observation from [20] will be useful in the proof.

Observation 7.1. *Let E be 3-lacunary, then the following holds for every $P \in \mathbf{P}_E$ and $Q \in E$: if $\frac{5}{4}\omega_P \cap \frac{5}{6}\omega_{3,Q} \neq \emptyset$ then $|I_P| \leq |I_Q|$.*

Proof. Assume the contrary, that is for some $P \in \mathbf{P}_E$ and $Q, R \in E$ it holds that $\frac{5}{4}\omega_P \cap \frac{5}{6}\omega_{3,Q} \neq \emptyset$, $\frac{5}{4}\omega_P \cap \frac{5}{6}\omega_{R_3} \neq \emptyset$, and $|I_R| \geq |I_P| \geq 2|I_Q|$. It follows that $|I_R| \geq 2|I_Q|$, therefore using sparseness of \mathbf{Q} it is clear that $5\omega_{R_3} \subset 5\omega_{Q_3}$, which contradicts the fact that E is 3-lacunary and \mathbf{Q} is sparse. \square

Now, fix a Schwarz function f_3 on \mathbb{R} . Let $a_j(Q) := \langle f_j, \phi_{j,Q} \rangle$ for $j = 1, 2$, and $d(P) = \langle f_3, \tilde{\phi}_P \rangle$ where $\tilde{\phi}_P := \sum_j \tilde{\phi}_{P,j} d_j$. Let K and its adjoint be defined by

$$\begin{aligned} (Kf)(Q) &:= \sum_{P \in \mathbf{P}: |I_P| \leq |I_Q|} |I_P| f(P) \langle \phi_P, \phi_{3,Q} \rangle, \quad Q \in \mathbf{Q}, \\ K^*f(P) &:= \sum_{Q \in \mathbf{Q}: |I_P| \leq |I_Q|} |I_Q| f(Q) \langle \phi_{3,Q}, \phi_P \rangle, \quad P \in \mathbf{P}. \end{aligned}$$

Now $\langle T_{BC}(f_1, f_2), f_3 \rangle = \sum_{Q \in \mathbf{Q}} |I_Q| a_1(Q) a_2(Q) \overline{(Kd)(Q)}$, which will be estimated using outer measure techniques.

7.1. Outer L^p estimates for K and K^* . Recall that \mathbf{Q} has rank 1. The following two Lemmas are the main estimates of this section.

Lemma 7.2. *Let $1 < q \leq 2$. For every $p \in (q', \infty)$ we have*

$$\|Kg\|_{\mathcal{L}^p(\mathbf{Q}, S_2^{[3]}, \sigma)} \lesssim \|g\|_{\mathcal{L}^p(\mathbf{P}, S_{q', lac} + S_{1, overlap}, \mu)}.$$

Lemma 7.3. *Let $q \in (1, 2]$. Then for any $1 < p < q$ we have*

$$\|K^*f\|_{\mathcal{L}^p(\mathbf{P}, S_{q, lac}, \mu)} \lesssim \|f\|_{\mathcal{L}^p(\mathbf{Q}, S_2^{[3]} + S_1^{[1,2]}, \sigma)}.$$

We will deduce Lemma 7.2 from Lemma 7.3 using a simple duality argument. By interpolation it suffices to consider weak-type estimates. Fix $\lambda > 0$ and $q' < p < \infty$. Without loss of generality assume $\|Kg\|_{\mathcal{L}^\infty(\mathbf{Q}, S_2^{[3]}, \sigma)} \leq 2\lambda$. Similar to the proof of Theorem 6.1, we may select¹ a strongly disjoint collection of 3-lacunary sets (E_m) and \mathbf{Q}' containing all E_m such that $\|Kg\|_{\mathcal{L}^\infty(\mathbf{Q} \setminus \mathbf{Q}', S_2^{[3]}, \sigma)} \leq \lambda$ and

$$\lambda^2 \sigma(\mathbf{Q}') \lesssim M := \sum_m \sum_{Q \in E_m} |I_Q| |Kg(Q)|^2.$$

Now, let $f(Q) = Kg(Q)$ for $Q \in \bigcup_m E_m$ and zero elsewhere, we obtain via applications of the outer Radon–Nikodym/Hölder inequalities

$$\begin{aligned} M &= \sum_{Q \in \mathbf{Q}} |I_Q| f(Q) \overline{(Kg)(Q)} = \sum_{P \in \mathbf{P}} |I_P| (K^*f)(P) \overline{g(P)} \\ &\lesssim \|K^*f\|_{\mathcal{L}^{p'}(\mathbf{P}, S_{q, lac}, \mu)} \|g\|_{\mathcal{L}^p(\mathbf{P}, S_{q', lac} + S_{1, overlap}, \mu)}. \end{aligned}$$

Therefore, by Lemma 7.3 we obtain

$$M \lesssim \|f\|_{\mathcal{L}^{p'}(\mathbf{Q}, S_2^{[3]} + S_1^{[1,2]}, \sigma)} \|g\|_{\mathcal{L}^p(\mathbf{P}, S_{q', lac} + S_{1, overlap}, \mu)}.$$

Note that f is supported on $\mathbf{Q}'' := \bigcup E_m$. It is clear that any 3-overlapping subset of \mathbf{Q}'' is essentially an union of spatially disjoint tritiles. Therefore $S_1^{[1,2]}(f) \lesssim S_\infty(f) \lesssim S_2^{[3]}(f) \lesssim \lambda$. We obtain

$$M \lesssim \lambda \sigma(\mathbf{Q}')^{1/p'} \|g\|_{\mathcal{L}^p(\mathbf{P}, S_{q', lac} + S_{1, overlap}, \mu)}$$

¹For more details see the proof of the L^2 case of Lemma 7.4.

Collecting estimates we obtain the desired weak-type estimate

$$\lambda\sigma(\mathbf{Q}')^{1/p} \lesssim \|g\|_{\mathcal{L}^p(\mathbf{P}, S_{q', lac} + S_{1, overlap}, \mu)} \quad .$$

In the rest of the section, we prove Lemma 7.3.

Proof of Lemma 7.3. By interpolation, it suffices to prove weak-type estimates for any fixed $p \in [1, q)$. Without loss of generality, assume $\|f\|_{\mathcal{L}^p(\mathbf{Q}, S_2^{[3]} + S_1^{[1,2]}, \sigma)} = 1$.

Fix any $\lambda > 0$. We will show that there exists $\mathbf{P}' \subset \mathbf{P}$ such that $\mu(\mathbf{P}') \lesssim \lambda^p$ and $\|K^*f\|_{\mathcal{L}^\infty(\mathbf{P} \setminus \mathbf{P}', S_{q, lac}, \mu)} \leq \lambda$. Without loss of generality we may assume that

$$(21) \quad \|K^*f\|_{\mathcal{L}^\infty(\mathbf{P}, S_{q, lac}, \mu)} < 2\lambda \quad .$$

The construction of \mathbf{P}' is similar to the proof of Theorem 6.1, and we also obtain a strongly disjoint collection of lacunary sets $(E_m)_{m \geq 1}$ contained inside \mathbf{P}' with the following property:

$$\lambda^q \mu(\mathbf{P}') \leq \lambda^q \sum_m |I_{E_m}| \leq \sum_{P \in \bigcup E_m} |I_P| |(K^*f)(P)|^q \quad .$$

Let $h(P) := K^*f(P)|K^*f(P)|^{q-2}$ for $P \in \mathbf{P}'$ and zero elsewhere, and let M be the last right hand side, which could be rewritten as

$$M = \sum_{P \in \mathbf{P}} |I_P| (K^*f)(P) \overline{h(P)} = \sum_{Q \in \mathbf{Q}} |I_Q| f(Q) \overline{K(h)(Q)} \quad .$$

By the classical Hölder inequality we have

$$S_1(fg) \lesssim S_2^{[3]}(f)S_2^{[3]}(g) + S_1^{[1,2]}(f)S_\infty(g) \lesssim (S_2^{[3]} + S_1^{[1,2]})(f)S_2^{[3]}(g) \quad .$$

Using outer Radon–Nikodym, outer Hölder, the normalization $\|f\|_{\mathcal{L}^p} = 1$, we have

$$M \lesssim \|f Kh\|_{L^1(\mathbf{Q}, S_1, \sigma)} \lesssim \|Kh\|_{L^{p'}(\mathbf{Q}, S_2^{[3]}, \sigma)} \quad .$$

Using Lemma 7.4 and (21) and the definition for h , it follows that

$$\|Kh\|_{L^{p'}(\mathbf{Q}, S_2^{[3]}, \sigma)} \lesssim \lambda^{q/q'} \left(\sum_k |I_{E_k}| \right)^{1/p'} \lesssim \lambda^{q/q' - q/p'} M^{1/p'} \quad .$$

Collecting estimates we obtain $M^{1/p} \lesssim \lambda^{q/q' - q/p'}$, therefore

$$\mu(\mathbf{P}')^{1/p} \lesssim \lambda^{-q/p} M^{1/p} \lesssim \lambda^{-1} \quad . \quad \square$$

Lemma 7.4. *Let $q \in (1, 2]$ and (E_k) be strongly disjoint lacunary in \mathbf{P} . Then for $q_1 \in (q', \infty]$ and every $h : \mathbf{P} \rightarrow \mathbb{C}$ that vanishes outside $\mathbf{P}' := \bigcup E_k$ it holds that*

$$\|Kh\|_{\mathcal{L}^{q_1}(\mathbf{Q}, S_2^{[3]}, \sigma)} \lesssim \|h\|_{\mathcal{L}^\infty(\mathbf{P}_1, S_{q', lac}, \mu_1)} \left(\sum_k |I_{E_k}| \right)^{1/q_1}$$

and weak type estimates hold at $q_1 = q'$.

Remark: While Lemma 7.4 is weaker than Lemma 7.2, it will be directly proved as part of the proof of Lemma 7.3.

Proof. It suffices to prove the following stronger estimate, which holds for $q_1 > 2$:

$$(22) \quad \|Kh\|_{\mathcal{L}^{q_1}(\mathbf{Q}, S_2^{[3]}, \sigma)} \lesssim \left(\sum_{P \in \mathbf{P}} |I_P| |h(P)|^{q_1} \right)^{1/q_1} + \sup_{P \in \mathbf{P}} |h(P)| \left(\sum_k |I_{E_k}| \right)^{1/q_1} .$$

Clearly, if $q_1 \geq q'$ then the desired conclusion follows from (22).

By interpolation, it suffices to prove weak-type estimates at $q_1 = 2$ and $q_1 = \infty$. Since the right hand side of (22) is not technically an L^p norm, we will detail the interpolation argument. Assume that the weak type estimates hold at $q_1 = 2$ and $q_1 = \infty$. Let $2 < q_1 < \infty$, for each $\lambda > 0$ we decompose $h(P) = h_1(P) + h_2(P)$ where $h_1(P) = h(P)1_{|h(P)| \leq \lambda/C}$ and C is sufficiently large but about the size of the norm for the assumed case $q_1 = \infty$ of (22). Using sublinearity of K , it follows that

$$\sigma(S_2^{[3]}(Kh) > \lambda) \leq \sigma(S_2^{[3]}(Th_2) \gtrsim \lambda) .$$

By the assumed L^2 case of (22), the last display is bounded above by

$$\begin{aligned} & \lesssim \lambda^{-2} \left(\sum_{P \in \mathbf{P}} |I_P| |h(P)|^2 1_{\lambda < C|h(P)|} + \sup_P (|h(P)|^2 1_{\lambda < C|h(P)|}) \left(\sum_k |I_{E_k}| \right) \right) \\ & = \lambda^{-2} \left(\sum_{P \in \mathbf{P}} |I_P| |h(P)|^2 1_{\lambda < C|h(P)|} + \sup_P |h(P)|^2 1_{\lambda < C \sup_P |h(P)|} \left(\sum_k |I_{E_k}| \right) \right) \end{aligned}$$

here in the second term we are able to move the sup inside because for any $a \geq 0$ $g(x) := x 1_{a < x}$ is increasing for $x \in [0, \infty)$. Now, multiplying both side with λ^{q_1-1} and integrate over $\lambda \in (0, \infty)$ we will obtain the desired estimate. This completes the interpolation argument.

Case 1: $q_1 = \infty$. By a standard characterization for S_2 (see e.g. [20, Lemma 6.4]) it suffices to show that if $E \subset \mathbf{Q}$ is 3-lacunary then

$$(23) \quad \frac{1}{|I_E|} \left\| \left(\sum_{Q \in E} |Kh(Q)|^2 1_{I_Q} \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R})} \lesssim \sup_{P \in \mathbf{P}} |h(P)| .$$

Since \mathbf{Q} is very sparse, it is clear that for any interval I there is at most one $Q \in E$ such that $I_Q = I$. This remark will be used implicitly below.

Let \mathbf{P}_E be defined as in Observation 7.1 and let $\mathbf{P}'_E = \mathbf{P}' \cap \mathbf{P}_E$. It follows that $Kh(Q) = \langle \phi_{3,Q}, h_E \rangle$ where $h_E(x) := \sum_{P \in \mathbf{P}'_E} |I_P| \overline{h(P)} \phi_P(x)$, for every $Q \in E$. Also, since (E_k) is strongly disjoint lacunary, it is clear that the intervals $\{I_P, P \in \mathbf{P}'_E\}$ are pairwise disjoint.

Now, by a standard argument (see e.g. [20, Lemma 6.8]), it follows that

$$\frac{1}{|I_E|} \left\| \left(\sum_{Q \in E} |Kh(Q)|^2 1_{I_Q} \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R})} \lesssim_N \frac{1}{|I_E|} \int \tilde{\chi}_{I_E}(x)^N |h_E(x)| dx .$$

Decompose $\mathbf{P}'_E = \mathbf{P}'_{E,1} \cup \mathbf{P}'_{E,2}$ where $\mathbf{P}'_{E,1} = \{P \in \mathbf{P}'_E : I_P \cap 3I \neq \emptyset\}$. It suffices to show that for every interval I of the same length as I_E it holds for $j = 1, 2$ that

$$A_j := \frac{1}{|I|} \int_I \left| \sum_{P \in \mathbf{P}'_{E,j}} |I_P| h(P) \phi_P(x) \right| dx \lesssim \sup_P |h(P)| .$$

For A_1 , notice first that for every $P \in \mathbf{P}'_{E,1}$ we have $I_P \subset 5I$. Since ϕ_P is L^1 -normalized and $\{I_P, P \in \mathbf{P}'_{E,1}\}$ are disjoint, we obtain

$$A_1 \lesssim \frac{1}{|I|} \sum_{P \in \mathbf{P}'_{E,1}} |I_P| |h(P)| \lesssim \sup_{P \in \mathbf{P}} |h(P)| .$$

To estimate A_2 , we decompose the summation over $\mathbf{P}'_{E,2}$ according to the length of I_P . Recall that $\{I_P : P \in \mathbf{P}'_{E,2}\}$ are disjoint and disjoint from $3I$. The desired estimate for A_2 follows from the following pointwise estimate on I :

$$\sum_{P \in \mathbf{P}'_{E,2}} |I_P| |\phi_{P_1}(x)| \lesssim \sum_n \sum_{P: |I_P|=2^n} \left(\frac{|I_P|}{|I|} \right)^4 \lesssim 1 .$$

Case 2: $q_1 = 2$. Fix $\lambda > 0$. We need to show existence of $\mathbf{Q}' \subset \mathbf{Q}$ such that

$$(24) \quad \lambda^2 \sigma(\mathbf{Q}') \lesssim N := \sum_{P \in \mathbf{P}'} |I_P| |h(P)|^2 + \sup_{P \in \mathbf{P}'} |h(P)|^2 \sum |I_{E_k}|$$

and $\|Kh\|_{\mathcal{L}^\infty(\mathbf{Q} \setminus \mathbf{Q}')} \leq \lambda$. Without loss of generality, assume that $\|Kh\|_{\mathcal{L}^\infty(\mathbf{Q})} \leq 2\lambda$.

Now, we will construct $\mathbf{Q}' = \mathbf{Q}'_1 \cup \mathbf{Q}'_2$ where \mathbf{Q}'_j is union of a collection \mathbf{S}_j of strongly disjoint j -lacunary sets, to be selected below. We first collect \mathbf{S}_1 using the following algorithm:

(i) If there exists $G \subset \mathbf{Q}$ is 1-lacunary and $(\frac{1}{|G|} \sum_{Q \in G} |I_Q| |Kh(Q)|^2)^{1/2} > \lambda$, we select one such G_1 with smallest possible $c(\omega_{Q_G})$.

(ii) Remove all $Q \in \mathbf{Q}$ such that for some $j = 1, 2, 3$ we have $Q_j \leq Q_{G_1, j}$.

We repeat the above argument and continue selecting G_2, G_3, \dots . Since \mathbf{Q} is finite the selection argument will stop. By geometry, (G_j) is strongly disjoint. Let \mathbf{Q}'_1 be the set of all tritiles removed from \mathbf{Q} .

We similarly collect \mathbf{S}_2 a collection of strongly disjoint 2-lacunary sets, the difference is we maximize $c(\omega_{Q_E})$ in step (i). Without loss of generality, assume that $\mu(\mathbf{Q}') \lesssim \mu(\mathbf{Q}'_1)$, which we will estimate below.

Let $k(Q) = Kh(Q)$ for $Q \in \mathbf{Q}'_1$ and zero elsewhere, clearly

$$\lambda^2 \mu(\mathbf{Q}') \lesssim \lambda^2 \mu(\mathbf{Q}'_1) \lesssim \lambda^2 \sum_j |I_{G_j}| \lesssim M ,$$

$$M := \sum_{Q \in \mathbf{Q}'_1} |I_Q| |k(Q)|^2 = \sum_{P \in \mathbf{P}'} \sum_{Q \in \mathbf{Q}'_1} 1_{|I_P| \leq |I_Q|} |I_Q| |I_P| \overline{k(Q)} h(P) \langle \phi_P, \phi_{3,Q} \rangle .$$

Now, to show (24) for \mathbf{Q}'_1 it suffices to show that

$$(25) \quad M \lesssim N^{1/2} M^{1/2} .$$

We first estimate the analogous double sum M_{free} where we don't include the coupling condition $|I_P| \leq |I_Q|$. By Cauchy-Schwarz and Lemma 6.1 and the assumptions $\|Kh\|_{\mathcal{L}^\infty(\mathbf{Q}, S_2^{[3]}, \mu)} \lesssim \lambda$, it follows that

$$M_{free} \lesssim \left\| \sum_{P \in \mathbf{P}'} |I_P| h(P) \phi_P \right\|_2 \left\| \sum_{Q \in \mathbf{Q}'_1} |I_Q| k(Q) \phi_{3,Q} \right\|_2 \lesssim N^{1/2} M^{1/2} .$$

We now consider the diagonal sum M_{diag} where $|I_P| = |I_Q|$. We may further assume that $|I_P| + \text{dist}(I_Q, I_P) \sim 2^n |I_Q|$ provided that there is an extra decaying factor in the estimate. Note that in order for $\langle \phi_P, \phi_{3,Q} \rangle$ to be nonzero the frequency support of ϕ_P and $\phi_{3,Q}$ must overlap. Thus essentially Q is determined from P and vice versa. Using $|\langle \phi_P, \phi_{3,Q} \rangle| \lesssim 2^{-n} |I_P|^{-1/2} |I_Q|^{-1/2}$ and Cauchy-Schwarz,

$$M_{diag} \lesssim 2^{-n} \left(\sum_{P \in \mathbf{P}'} |I_P| |h(P)|^2 \right)^{1/2} \left(\sum_{Q \in \mathbf{Q}'_1} |I_Q| |k(Q)|^2 \right)^{1/2} \lesssim 2^{-n} N^{1/2} M^{1/2} .$$

Thus, to prove (25) the roles of \mathbf{P}' and \mathbf{Q}'_1 are fairly symmetric. We will assume below that $\sum_m |I_{G_m}| \leq \sum_k |I_{E_k}|$, the proof for the other case is entirely similar. Using Cauchy Schwarz, it suffices to show that for every $G \in \{G_1, G_2, \dots\}$ we have

$$\begin{aligned} M_G &:= \sum_{Q \in G} \sum_{P \in \mathbf{P}'} 1_{|I_P| \leq |I_Q|} |I_P| |I_Q| h(P) \overline{k(Q)} \langle \phi_P, \phi_{3,Q} \rangle \\ (26) \quad &\lesssim \sup_P |h(P)| \left(\left(\sum_{Q \in G} |k(Q)|^2 |I_Q| \right)^{1/2} |I_G|^{1/2} + \sup_Q |k(Q)| |I_G| \right) . \end{aligned}$$

Now, similar to Observation 7.1, we may write

$$M_G = \sum_{Q \in G} |I_Q| \overline{k(Q)} \langle h_G, \phi_{3,Q} \rangle , \quad h_G := \sum_{P \in \mathbf{P}'_G} h(P) |I_P| \phi_P ,$$

and \mathbf{P}'_G contains all $P \in \mathbf{P}'$ such that for some $Q \in G$ we have $\frac{5}{4}\omega_P \cap \frac{5}{6}\omega_{3,Q} \neq \emptyset$ and $|I_P| \leq |I_Q|$. Using strong disjointness, it is clear that the intervals of the elements of \mathbf{P}'_G are essentially pairwise disjoint.

We now estimate the contribution of those $P \in \mathbf{P}'_G$ such that $I_P \cap 4I_G \neq \emptyset$. Clearly we will have $I_P \subset 6I_G$. Let $h_{E,0}$ be the corresponding subsum of h_G . By Cauchy Schwarz and standard Calderon-Zygmund theory, we have

$$\begin{aligned} \sum_{Q \in G} |I_Q| \overline{k(Q)} \langle h_{G,0}, \phi_{3,Q} \rangle &\lesssim \left(\sum_{Q \in G} |I_Q| |k(Q)|^2 \right)^{1/2} \left(\sum_{P \in \mathbf{P}'_G: I_P \subset 6I_G} |I_P| |h(P)|^2 \right)^{1/2} \\ &\lesssim \left(\sum_{Q \in G} |I_Q| |k(Q)|^2 \right)^{1/2} |I_G|^{1/2} \sup_P |h(P)| , \end{aligned}$$

in the last estimate we used disjointness of the intervals of elements of \mathbf{P}'_G .

Consider the contribution of other P 's. Let $\mathbf{P}'_{G,k}$ contains all $P \in \mathbf{P}'_G$ such that $I_P \cap 2^{k+2}I_G \neq \emptyset$ but $I_P \cap 2^{k+1}I_G = \emptyset$. Let $\Lambda := \sup_Q |k(Q)| \sup_P |h(P)|$. It suffices to show that for any $Q \in G$ we have

$$\sum_{P \in \mathbf{P}'_{G,k}: |I_P| \leq |I_Q|/C} |I_Q| |I_P| \overline{k(Q)} h(P) \langle \phi_P, \phi_{3,Q} \rangle \lesssim 2^{-k} \Lambda \left(\frac{|I_Q|}{|I_G|} \right)^2 |I_G|$$

We decompose $\phi_{3,Q} = \phi_{3,Q} 1_{2^k I_G} + \phi_{3,Q} (1 - 1_{2^k I_G})$. Since $k(Q)h(Q) \lesssim \Lambda$ and since I_P 's are essentially pairwise disjoint and contained in $2^{k+3}I_G \setminus 2^k I_G$, the left hand

side of the above display is bounded above by

$$\begin{aligned}
&\lesssim \Lambda \sum_P |I_P| |I_Q| \left(\|\phi_P 1_{2^k I_G}\|_1 \|\phi_{3,Q}\|_\infty + \|\phi_P\|_1 \|\phi_{3,Q} 1_{\mathbb{R} \setminus 2^k I_G}\|_\infty \right) \\
&\lesssim 2^{-2k} \Lambda \sum_{P: |I_P| \leq |I_Q|} |I_P| |I_Q| \left(\left(\frac{|I_P|}{2^k |I_G|} \right)^2 \frac{1}{|I_Q|} + \frac{1}{|I_Q|} \left(\frac{|I_Q|}{2^k |I_G|} \right)^2 \right) \\
&\lesssim 2^{-2k} \Lambda \left(\frac{|I_Q|}{|I_G|} \right)^2 \sum_{P: I_P \subset 2^{k+3} I_G} |I_P| \lesssim 2^{-k} \Lambda \left(\frac{|I_Q|}{|I_G|} \right)^2 |I_G| \quad . \quad \square
\end{aligned}$$

7.2. Outer L^∞ estimate for Kd . When $d(P) = \langle f_3, \tilde{\phi}_P \rangle$, we have

Lemma 7.5. *For any $\epsilon > 0$ we have*

$$\|Kd\|_{\mathcal{L}^\infty(\mathbf{Q}, S_2^{[3]}, \sigma)} \lesssim_{\epsilon, N} \sup_{Q \in \mathbf{Q}} \left(\frac{1}{|I_Q|} \int \tilde{\chi}_{I_Q}^N |f_3|^{1+\epsilon} \right)^{1/(1+\epsilon)} .$$

Proof. Let $E \subset \mathbf{Q}$ be 3-lacunary and define \mathbf{P}_E as in Observation 7.1. It follows that $Kd(Q) = \langle T_{\mathbf{P}_E}^* f_3, \phi_{3,Q} \rangle$ where $T_{\mathbf{P}_E}^*$ is the adjoint of

$$T_{\mathbf{P}_E} f := \sum_j \sum_{P \in \mathbf{P}_E} |I_P| \langle f, \phi_P \rangle \tilde{\phi}_{P,j} d_j .$$

Therefore, similar to the $L^{2,\infty}$ case of Lemma 7.3, it suffices to show that

$$\frac{1}{|I|} \int_I |(T_{\mathbf{P}_E}^* f_3)(x)| dx \lesssim \left(\frac{1}{|I|} \int \tilde{\chi}_I^N |f_3|^{1+\epsilon} \right)^{1/(1+\epsilon)} ,$$

for every interval I of the same length as I_E .

Now, it is clear that \mathbf{P}_E is an overlapping generating subset of \mathbf{P} . It follows that for any x only one j and one scale of \mathbf{P}_E would contribute to the defining summation of $T_{\mathbf{P}_E} f$. Thus, $T_{\mathbf{P}_E} f$ is controlled by the maximal function of $\sum_{P \in \mathbf{P}_E} |I_P| \langle f, \phi_P \rangle \phi_P$ and so by Calderon–Zygmund theory, $T_{\mathbf{P}_E}$ is bounded on $L^p(\mathbb{R})$ for any $1 < p < \infty$. By duality $T_{\mathbf{P}_E}^*$ is also bounded on $L^p(\mathbb{R})$. Furthermore, we also have

$$(27) \quad \langle T_{\mathbf{P}_E}^* g, h \rangle \lesssim \sum_{P \in \mathbf{P}_E} \frac{1}{|I_P|} \langle |g|, \tilde{\chi}_P^N \rangle \langle \tilde{\chi}_{I_P}(x)^N, |h| \rangle \quad , \quad N > 0 \quad ,$$

from there we obtain a pointwise estimate for $T_{\mathbf{P}_E}^* g(x)$ which holds for a.e. x .

Decompose $f_3 = f_3 1_{7I} + f_3 1_{(7I)^c}$. By boundedness of $T_{\mathbf{P}_E}^*$ and Hölder inequality, the contribution of $f_3 1_{7I}$ could be easily controlled. For the contribution of $f_3 1_{(7I)^c}$, let $\mathbf{P}_{E,1} = \{P \in \mathbf{P}_E : I_P \subset 5I\}$, and $\mathbf{P}_{E,2} = \mathbf{P}_E \setminus \mathbf{P}_{E,1}$. In $\mathbf{P}_{E,1}$ clearly $\tilde{\chi}_{I_P} \lesssim \tilde{\chi}_I$, thus using the resulting pointwise estimate resulting from (27) we obtain

$$T_{\mathbf{P}_{E,1}}^* (f_3 1_{(7I)^c})(x) \lesssim \sum_{P \in \mathbf{P}_{E,1}} |I_P|^{-1} \left\langle |f_3|, \left(\frac{|I_P|}{|I|} \right)^2 \tilde{\chi}_I^N \right\rangle \tilde{\chi}_{I_P}(x)^2 \lesssim \frac{1}{|I|} \langle |f_3|, \tilde{\chi}_I^N \rangle .$$

For $\mathbf{P}_{E,2}$ it is clear that $\tilde{\chi}_{I_P}(x)\tilde{\chi}_{I_P}(y) \leq (|x-y|/|I_P|)^{-2} \lesssim \tilde{\chi}_I(y)$ for every $x \in I$ and $y \in (7I)^c$. It follows that

$$\begin{aligned} \|T_{\mathbf{P}_{E,2}}^* f_3\|_{L^1(I)} &\lesssim \int |f_3(y)| \tilde{\chi}_I(y)^N \sum_P \frac{1}{|I_P|} \tilde{\chi}_{I_P}(y)^2 \left(\frac{|I_P|}{|I|}\right)^{-2} \|\tilde{\chi}_{I_P}\|_{L^1(I)} dy \\ &\lesssim \int |f_3(y)| \tilde{\chi}_I(y)^N \sum_P \tilde{\chi}_{I_P}(y)^2 \left(\frac{|I_P|}{|I|}\right)^2 dy \\ &\lesssim \int |f_3(y)| \tilde{\chi}_I(y)^N dy \quad . \quad \square \end{aligned}$$

7.3. Proof of Theorem 7.1.

7.3.1. *The basic range.* Let $q = \min(2, r/2) = r/2 \in (1, 2)$. Let $a_3 = \overline{Kd}$. Assume $q_1, q_2, q_3 \geq 1$ such that $\sum 1/q_j = 1$. By classical Hölder, we obtain

$$S_1(a_1 a_2 a_3) \lesssim S_2^{[1]}(a_1) S_2^{[2]}(a_2) S_2^{[3]}(a_3) \quad .$$

Using outer Radon-Nikodym/Hölder inequalities, it follows that

$$(28) \quad \Lambda_{\mathbf{P}, \mathbf{Q}} \lesssim \prod_{j=1,2,3} \|a_j\|_{\mathcal{L}^{q_j}(\mathbf{Q}, S_2^{[j]}, \sigma)} \quad ,$$

provided that $\sum 1/q_j = 1$ and $q_3 > q'$. If $q_1, q_2 > 2$ then via Lemma 7.2 and Carleson embeddings (Theorem 6.1, Theorem 6.5) we obtain $\Lambda_{\mathbf{P}, \mathbf{Q}} \lesssim \prod_j \|f_j\|_{q_j}$.

7.3.2. *Extending the range.* To extend the range, we will use restricted weak-type interpolation, following [22]. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be such that $\sum_j \alpha_j = 1$ and $\alpha_j \geq 0$. Then we say that a trilinear form $\Lambda(f_1, f_2, f_3)$ satisfies restricted weak-type estimates with exponents α if the following holds: there exists $j_0 \in \{1, 2, 3\}$ such that for every $F_1, F_2, F_3 \subset \mathbb{R}$ finite Lebesgue measures we could find $B \subset F_{j_0}$ with less than half of the measure, so that $|\Lambda_{\mathbf{P}, \mathbf{Q}}(f_1, f_2, f_3)| \lesssim |F_1|^{\alpha_1} |F_2|^{\alpha_2} |F_3|^{\alpha_3}$ whenever $|f_j| \leq 1_{F_j}$ for every j and furthermore $|f_{j_0}| \leq 1_{F_{j_0}-B}$. When exactly one of the index is negative, say $\alpha_k < 0$, we say that $\Lambda(f_1, f_2, f_3)$ satisfies restricted weak-type estimates with exponents α if the previous claim holds with $j_0 = k$.

From [22], if $T(f_1, f_2)$ is a bilinear operator such that $\langle T(f_1, f_2), f_3 \rangle$ satisfies restricted weak-type estimates for a finite collection V of triples then

$$\|T(f_1, f_2)\|_{p_3} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}$$

for any triples of exponents $0 < p_1, p_2, p_3 \leq \infty$ such that $(1/p_1, 1/p_2, 1/p_3)$ is in the interior of the convex hull of V .

Thus, to show Theorem 7.1, it suffices to show that $\Lambda_{\mathbf{P}, \mathbf{Q}}(f_1, f_2, f_3)$ satisfies Let $\mathbf{H} \subset \{\alpha_1 + \alpha_2 + \alpha_3 = 1\}$ with vertices:

$$\begin{aligned} &H_1\left(\frac{1}{2}, -\frac{1}{2}, 1\right) \quad , \quad H_2\left(-\frac{1}{2}, \frac{1}{2}, 1\right) \quad , \\ &H_3\left(\frac{1}{2} - \frac{2}{q'}, \frac{1}{2} + \frac{1}{q'}, \frac{1}{q'}\right) \quad , \quad H_4\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{q'}, -\frac{1}{q'}\right) \quad , \\ &H_5\left(\frac{1}{2} + \frac{1}{q'}, \frac{1}{2}, -\frac{1}{q'}\right) \quad , \quad H_6\left(\frac{1}{2} + \frac{1}{q'}, \frac{1}{2} - \frac{2}{q'}, \frac{1}{q'}\right) \quad . \end{aligned}$$

Our proof below will be fairly symmetric accross the vertices, so we will show the claim only for H_2 and H_3 . For any $\epsilon > 0$ and $q_1, q_2 > 2$, $q_3 \geq q' + \epsilon$, it follows from (28), Carleson embeddings, and convexity that

$$(29) \quad \Lambda_{\mathbf{P}, \mathbf{Q}} \lesssim \|a_1\|_\infty^{1-\frac{2}{q_1}} \|a_2\|_\infty^{1-\frac{2}{q_2}} \|a_3\|_\infty^{1-\frac{q'+\epsilon}{q_3}} \prod_{j=1,2,3} |F_j|^{1/q_j} .$$

By Lemma 6.3, for $j = 1, 2$ we have

$$(30) \quad \|a_j\|_{\mathcal{L}^\infty(\mathbf{Q}, S_2^{[j]}, \sigma)} \lesssim_N \sup_{Q \in \mathbf{Q}} \frac{1}{|I_Q|} \int \tilde{\chi}_{I_Q}(x)^N |f_j(x)| dx .$$

For $\|a_3\|_\infty$ we will use Lemma 7.5 and obtain

$$(31) \quad \|a_3\|_{\mathcal{L}^\infty(\mathbf{Q}, S_2^{[3]}, \sigma)} \lesssim_N \sup_{Q \in \mathbf{Q}} \left(\frac{1}{|I_Q|} \int \tilde{\chi}_{I_Q}(x)^N |f_3(x)|^{1+\epsilon} dx \right)^{\frac{1}{1+\epsilon}} .$$

We now consider neighborhood of H_1 and H_2 . Near these vertices we have $\alpha_1 < 0$, so we may choose $G_1 = F_1 \setminus B$ where $B = \bigcup_{j=2}^3 \{M(1_{F_j}) > C|F_j|/|F_1|\}$ and C is large enough to ensure $|B| < |F_1|/2$.

Without loss of generality assume that $2^m \leq 1 + \text{dist}(I_Q, B^c)/|I_Q| < 2^{m+1}$ for some $m \geq 0$ integer, provided that we could obtain extra decaying factors.

From (30) we obtain $\|a_2\|_{\mathcal{L}^\infty} \lesssim 2^m \sup_{x \in B^c} M1_{F_2}(x) \lesssim 2^m |F_2|/|F_1|$ and similarly using (31) we have $\|a_3\|_{\mathcal{L}^\infty} \lesssim_\epsilon (2^m |F_2|/|F_1|)^{1/(1+\epsilon)}$. It follows from (29) that

$$\Lambda_{\mathbf{P}, \mathbf{Q}} \lesssim 2^{-m} |F_1|^{\alpha_1} |F_2|^{\alpha_2} |F_3|^{\alpha_3}$$

$$\alpha_1 = \frac{1}{q_1} + \frac{2}{q_2} + \frac{q' + \epsilon}{q_3(1 + \epsilon)} - 1 - \frac{1}{1 + \epsilon} , \quad \alpha_2 = 1 - \frac{1}{q_2} , \quad \alpha_3 = \frac{1}{1 + \epsilon} - \frac{q' + \epsilon}{q_3(1 + \epsilon)} + \frac{1}{q_3} .$$

Thus, letting (ϵ, q_1, q_3) close to $(0, 2, q')$, we can make α arbitrarily close to H_3 . Similarly, letting (ϵ, q_1, q_3) close to $(0, 2, \infty)$ we can make α arbitrarily close to H_2 . This completes the proof of Theorem 7.1.

8. ESTIMATES FOR CC MODEL OPERATORS

Theorem 8.1. *Let T be a CC model operator. Then $\|T\|_{L^{q_1} \times L^{q_2} \rightarrow L^{q_3}} < \infty$ for every $1/q = 1/q_1 + 1/q_2$ such that $\frac{2r}{3r-4} < q_1, q_2 < \infty$ and $q_3 > \frac{r'}{2}$.*

For simplicity, we will assume that $\omega_{j,P,upper}$ are finite intervals and $\omega_{j,P,lower}$ are halflines for $j = 1, 2$; the other cases are either symmetric or could be reduced to this situation. We will use the same set up for outer measures space in Section 6.1 for both \mathbf{P}_1 and \mathbf{P}_2 . For convenience of notation, let μ_1 and μ_2 be the corresponding outer measures.

For every x , $P \in \mathbf{P}_1$, and $P' \in \mathbf{P}_2$, there is at most one $1 \leq j \leq K(x)$ such that

$$N_{j-1} \in \omega_{1,P,lower} \cap \omega_{2,P',lower} , \quad N_j \in \omega_{1,P,upper} \cap \omega_{2,P',upper} .$$

Let $d_{P,P'}(x) = d_j(x)$ if such j exists, and zero otherwise; define d_P and $d_{P'}$ similarly.

Fix f_3 Schwartz on \mathbb{R} and let $a_1(P) = \langle f_1, \phi_{1,P} \rangle$ and $a_2(P') = \langle f_2, \phi_{2,P'} \rangle$, and

$$\begin{aligned} Kf(P) &:= \sum_{P' \in \mathbf{P}_2: |I_P| \geq 16|I_{P'}|} |I_{P'}| f(P') \langle \phi_{1,P} \phi_{2,P'} d_{P,P'}, f_3 \rangle \\ K^*g(P') &:= \sum_{P \in \mathbf{P}_1: |I_P| \geq 16|I_{P'}|} |I_P| g(P) \langle \phi_{1,P} \phi_{2,P'} d_{P,P'}, f_3 \rangle . \end{aligned}$$

Clearly, $\langle T(f_1, f_2), f_3 \rangle = \sum_{P \in \mathbf{P}_1} |I_P| a_1(P) (K a_2)(P)$. To prove Theorem 8.1 we first establish outer L^p estimates for K .

For convenience of notation, assume that f_3 is supported on a fixed set F_3 . Let $b_P(x) := f_3 d_j T_{j, |I_P|/16} f$ if there exists (a unique) j such that $N_{j-1} \in \omega_{1,P, \text{lower}}$ and $N_j \in \omega_{1,P, \text{upper}}$, and let $b_P(x) = 0$ otherwise. Also, define

$$M_N(F_3) := M_N(\mathbf{P}_1, 1_{F_3})^{1/p_1} M_N(\mathbf{P}_2, 1_{F_3})^{1/p_2} .$$

(Recall the definition of M_N from (14).)

8.1. Outer L^p estimates for K . The following Lemma is the main estimate of the current section, and we will always assume that $1/p_1 + 1/p_2 + 1/p_3 = 1$ and $2 < p_1, p_2 < 2r/(4-r)$, and $\infty > p_3 > r/(r-2)$. All implicit constants may depend on these exponents.

Lemma 8.2. *It holds that*

$$\|Kf\|_{\mathcal{L}^{p'_1}(\mathbf{P}_1, S_{1, \text{overlap}} + S_{2, \text{lac}}, \mu_1)} \lesssim_N M_N(F_3) \|f\|_{\mathcal{L}^{p_2}(\mathbf{P}_2, S_{2, \text{lac}}, \mu_2)} \|f_3\|_{p_3} .$$

The proof of Lemma 8.2 consists of two parts. The factor $M_N(F_3)$ should be thought of an estimate for the norm of $K : \mathcal{L}^{p_2}(\mathbf{P}_2) \times L^{p_3}(F_3) \rightarrow \mathcal{L}^{p_1}(\mathbf{P}_1)$, capturing the interaction of $\text{supp}(f)$ (i.e. \mathbf{P}_2), $\text{supp}(Kf)$ (i.e. \mathbf{P}_1) and $\text{supp}(f_3)$. Thus, by interpolation (Lemma 5.2) it suffices to show the weak-type estimates (note that the constant $M_N(F_3)$ could be naturally absorbed inside the (outer)measures on the right hand side), and we will treat the contribution of $S_{1, \text{overlap}}$ in Section 8.1.1 and the contribution of $S_{2, \text{lac}}$ in Section 8.1.2.

We first fix some notations. For each j and any $f : \mathbf{P}_2 \rightarrow \mathbb{C}$ and $\alpha > 0$ let

$$T_{j, \alpha}(f) = \sum_{P' \in \mathbf{P}_2: |I_{P'}| \leq \alpha} |I_{P'}| f(P') \tilde{\phi}_{2, P', j} , \quad T_j^* f = \sup_{\alpha} |T_{j, \alpha} f| .$$

For every $E \subset \mathbf{P}_1$ let $\mathbf{P}_2(E)$ contains all $P' \in \mathbf{P}_2$ such that for some $P \in E$ we have $|I_{P'}| \leq |I_P|/16$ and $\frac{5}{4}\omega_{1, P_1, \text{upper}} \cap \frac{5}{4}\omega_{2, P', \text{upper}} \neq \emptyset$.

8.1.1. The overlapping setting. In this section we prove that

$$\|Kf\|_{\mathcal{L}^{p'_1, \infty}(\mathbf{P}_1, S_{1, \text{overlap}}, \mu_1)} \lesssim_N M_N(\mathbf{P}_1, 1_{F_3})^{1/p_1} \|f\|_{\mathcal{L}^{p_2}(\mathbf{P}_2, S_{2, \text{lac}}, \mu_2)} \|f_3\|_{L^{p_3}(\mathbb{R})} .$$

Using Hölder inequality, it follows from Lemma 8.3 that

$$\begin{aligned} &\|Kf\|_{\mathcal{L}^\infty(\mathbf{P}_1, S_{1, \text{overlap}}, \mu_1)} \\ &\lesssim_N M_N(\mathbf{P}_1, 1_{F_3})^{1/p_1} \sup_{R \in \overline{\mathbf{P}}_1} \left(\frac{1}{|I_R|} \int \tilde{\chi}_{I_R}^N \sum_{j: N_j \in \tilde{\omega}_R} |f_3 d_j T_j^*(f)|^{p'_1} dx \right)^{1/p'_1} \end{aligned}$$

Therefore, it follows from Lemma 6.6 that

$$\|Kf\|_{\mathcal{L}^{p'_1, \infty}(\mathbf{P}_1, S_{1, \text{overlap}}, \mu_1)} \lesssim_N M_N(\mathbf{P}_1, 1_{F_3})^{1/p_1} \|f_3\|_{p_1} \left(\sum_{1 \leq j \leq K} |d_j T_j^*(f)|^{p'_1} \right)^{1/p'_1}$$

so using $\sum_j |d_j|^{r/(r-2)} = O(1)$ it follows that

$$\|Kf\|_{\mathcal{L}^{p'_1, \infty}(\mathbf{P}_1, S_{1, \text{overlap}}, \mu_1)} \lesssim M_N(\mathbf{P}_1, 1_{F_3})^{1/p_1} \|f_3\|_{p_3} \left(\sum_{1 \leq j \leq K} |T_j^*(f)|^s \right)^{1/s} \|f_3\|_{L^{p_2}(F_3)} ,$$

where $s > 0$ such that $1/p'_1 \leq 1/s + (r-2)/r$. Since $1/p'_1 = 1/p_2 + 1/p_3 < 1/p_2 + (r-2)/r$, we could choose s such that $s > p_2$ (which is larger than 2). The desired estimate now follows from Theorem 6.9.

Lemma 8.3. *Uniform over $\mathbf{P} \subset \mathbf{P}_1$ it holds that*

$$\|K(f)\|_{\mathcal{L}^\infty(\mathbf{P}, S_{1, \text{overlap}}, \mu_1)} \lesssim_N m_{\infty, N}(\mathbf{P}, (f_3 d_j T_j^* f)) .$$

Proof. This follows from a simple adaptation of the proof of Lemma 6.8 and the fact that for every j and every P we have $|T_{j, |I_P|/16} f| \leq T_j^* f$. \square

8.1.2. *The lacunary setting.* In this section we prove that

$$\|Kf\|_{\mathcal{L}^{p'_1, \infty}(\mathbf{P}_1, S_{2, \text{lac}}, \mu_1)} \lesssim_N M_N(F_3) \|f\|_{\mathcal{L}^{p_2}(\mathbf{P}_2, S_{2, \text{lac}}, \mu_2)} \|f_3\|_{L^{p_3}(\mathbb{R})} .$$

Fix $\lambda > 0$. Without loss of generality assume that $\|Kf\|_{\mathcal{L}^\infty(S_{2, \text{lac}})} \leq 2\lambda$.

Apply a variant of the selection argument in the proof of Theorem 6.1 we may find $A \subset \mathbf{P}_1$ such that $\|Kf\|_{\mathcal{L}^\infty(\mathbf{P}_1 \setminus A, S_{2, \text{lac}}, \mu_1)} \leq \lambda$ and a strongly disjoint collection of generating subsets (E_m) covering A such that

$$\lambda^2 \mu_1(A) \lesssim \lambda^2 \sum_m |I_{E_m}| \lesssim M := \sum_{P \in E_m} |I_P| |Kf(P)|^2 .$$

Let $g(P) = \overline{Kf(P)}$ for $P \in \mathbf{Q}_1 := \bigcup E_m$ and $g(P) = 0$ otherwise. We obtain

$$M = \sum_{P \in \mathbf{Q}_1} |I_P| |Kf(P)|^2 g(P) = \sum_{P' \in \mathbf{P}_2} |I_{P'}| |f(P')| (K^* g)(P') ,$$

Using Lemma 8.4 and the assumption that $\|Kf\|_{\mathcal{L}^\infty(\mathbf{Q}_1, S_{2, \text{lac}}, \mu_1)} \leq 2\lambda$ we obtain

$$\|K^* g\|_{\mathcal{L}^{p'_2}(\mathbf{P}_2, S_{2, \text{lac}} + S_{1, \text{overlap}}, \mu_1)} \lesssim_N M_N(F_3) \lambda \left(\sum_m |I_{E_m}| \right)^{1/p_1} \|f_3\|_{p_3} ,$$

Using outer Radon-Nikodym/Hölder, it follows that

$$\begin{aligned} M &\lesssim \|f\|_{\mathcal{L}^{p_2}(\mathbf{P}_2, S_{2, \text{lac}}, \mu_2)} \|K^* g\|_{\mathcal{L}^{p'_2}(\mathbf{P}_2, S_{2, \text{lac}} + S_{1, \text{overlap}}, \mu_1)} \\ &\lesssim M_N(F_3) \lambda \|f\|_{\mathcal{L}^{p_2}(\mathbf{P}_2, S_{2, \text{lac}}, \mu_2)} \|f_3\|_{p_3} (\lambda^{-2} M)^{1/p_1} , \end{aligned}$$

from this the desired estimates for M (and hence for $\mu_1(A)$) easily follow.

Lemma 8.4. *It holds that (with $\|g\|_\infty = \|g\|_{\mathcal{L}^\infty(\mathbf{Q}_1, S_{2, \text{lac}}, \mu_1)}$)*

$$\|K^* g\|_{\mathcal{L}^{p'_2}(\mathbf{P}_2, S_{1, \text{overlap}} + S_{2, \text{lac}}, \mu_2)} \lesssim_N M_N(F_3) \|f_3\|_{p_3} \|g\|_\infty \left(\sum_m |I_{E_m}| \right)^{1/p_1} .$$

Proof. By simple modifications of the argument in Section 8.1.1 together with Theorem 6.9 part (i), we obtain

$$\begin{aligned} \|K^*g\|_{\mathcal{L}^{p'_2}(\mathbf{P}_2, S_{1,overlap}, \mu_2)} &\lesssim M_N(\mathbf{P}_2, 1_{F_3})^{1/p_2} \|f_3\|_{p_3} \|V_{p_1}(g)1_{F_3}\|_{p_1} \\ &\lesssim M_N(\mathbf{P}_1, 1_{F_3})^{1/p_1} M_N(\mathbf{P}_2, 1_{F_3})^{1/p_2} \|f_3\|_{p_3} \|g\|_{\mathcal{L}^{p_1}(S_{2,lac})} \quad , \end{aligned}$$

so it remains to consider the contribution of $S_{2,lac}$, and by interpolation it suffices to consider weak-type estimates.

Fix $\alpha > 0$. Without loss of generality we may assume that $\|K^*g\|_{\mathcal{L}^\infty(S_{2,lac})} \leq 2\alpha$. By a standard argument, we may find $B \subset \mathbf{P}_2$ with $\|K^*g\|_{\mathcal{L}^\infty(B, S_{2,lac}, \mu_2)} \leq \lambda$ and a collection of strongly disjoint lacunary generating sets (G_n) covering B such that

$$\alpha^2 \mu_2(B) \lesssim \alpha^2 \sum_n |I_{G_n}| \lesssim N := \sum_n \sum_{P' \in G_n} |I_{P'}| |K^*g(P')|^2 \quad .$$

Let $h(P') = \overline{K^*g(P')}$ for $P' \in \mathbf{Q}_2 = \bigcup G_n$ and let $h(P') = 0$ otherwise. We obtain

$$N = \sum_{\substack{P \in \mathbf{Q}_1, \quad P' \in \mathbf{Q}_2 \\ |I_P| \geq |I_{P'}|/16}} |I_P| |I_{P'}| h(P') g(P) \langle \phi_{1,P} \phi_{2,P'} d_{P,P'}, f_3 \rangle \quad ,$$

thus it suffices to show that

$$N \lesssim M_N(F_3) \|f_3\|_{p_3} \|h\|_{\mathcal{L}^\infty(S_{2,lac})} \|g\|_{\mathcal{L}^\infty(S_{2,lac})} \left(\sum_m |I_{E_m}| \right)^{\frac{1}{p_1}} \left(\sum_n |I_{G_n}| \right)^{\frac{1}{p_2}} \quad .$$

Note that since \mathbf{Q}_1 and \mathbf{Q}_2 are unions of strongly disjoint lacunary sets, all overlapping sets are essentially a collection of spatially disjoint tiles, therefore $S_{1,overlap} \lesssim S_{2,lac}$ in each of them. This observation will be used implicitly below.

We first show that the unconstrained double sum N_{free} over P, P' (where there is no constraint between P and P') satisfies the desired estimate. Indeed, since $2 < p_1, p_2 < 2r/(3r-4)$ and $1/p_1 + 1/p_2 = 1 - 1/p_3 > 2/r$ we may find $s, t > 2$ such that $2/r = 1/s + 1/t$ and $t > p_1$ and $s > p_2$. Using Theorem 6.9 we have

$$\begin{aligned} N_{free} &\lesssim_N M_N(\mathbf{P}_1, 1_{F_3})^{1/p_1} \|g\|_{\mathcal{L}^{p_1}(S_{2,lac})} \left\| \left(\sum_j \left| \sum_{P' \in \mathbf{Q}_1} |I_P| g(P) \tilde{\phi}_{2,P',j} d_j f_3 \right|^{t'} \right)^{1/t'} \right\|_{p'_1} \\ &\lesssim M_N(\mathbf{P}_1, 1_{F_3})^{1/p_1} \|g\|_{\mathcal{L}^{p_1}(S_{2,lac})} \|h\|_{\mathcal{L}^{p_2}(S_{2,lac})} M_N(\mathbf{P}_2, 1_{F_3})^{1/p_2} \|f_3\|_{p_3} \quad . \end{aligned}$$

We now consider diagonal sums when $|I_{P'}| = C|I_P|$ for some fixed $C \in [2^{-4}, 2^4]$. The proof below is symmetric in P, P' , so we will assume $C \leq 1$. We say that P is linked to P' if the corresponding summand is nonzero, clearly that all linked pairs satisfy $\omega_{1,P,upper} \cap \omega_{2,P',upper} \neq \emptyset$. Since these are dyadic intervals and $|\omega_{1,P,upper}| \leq |\omega_{2,P',upper}|$, we obtain $\omega_{1,P,upper} \subset \omega_{2,P',upper}$. Without loss of generality, assume that for some k it holds that $2^k |I_P| \leq |I_P| + \text{dist}(I_P, I_{P'}) < 2^{k+1} |I_P|$ for all linked pairs, provided that we have sufficient decay in the estimates. It follows that for each P' there is at most $O(1)$ linked P and vice versa, and by further dividing if necessary we may assume that exactly one P is linked to exactly one P' , and let $P' = F(P)$ and $P = F^{-1}(P')$. It follows that $h(P')$ and $c(P, P') := |I_{P'}| \langle \phi_{1,P} \phi_{2,P'} d_{P,P'}, f_3 \rangle$ are functions on \mathbf{P} . Using outer Radon–Nikodym/Hölder and the triangle inequalities, it follows that the corresponding sum is bounded by

$$N_{diag,k} \lesssim \|g\|_{\mathcal{L}^{p_1}(\mathbf{Q}_1, S_{2,\mu_1})} \|h \circ F\|_{\mathcal{L}^{p_2}(\mathbf{Q}_1, S_{2,\mu_1})} \|c\|_{\mathcal{L}^{p_3}(\mathbf{Q}_1, S_{\infty,\mu_1})} \quad .$$

Now, we note that since \mathbf{Q}_1 is an union of strongly disjoint lacunary sets, all the overlapping generating subsets of \mathbf{Q}_1 has $O(1)$ elements, therefore $S_2 \lesssim S_{2,lac}$ on \mathbf{Q}_1 . For any generating set $E_2 \subset \mathbf{Q}_2$ it is clear that $F^{-1}(E_2)$ could be covered by $O(1)$ generating sets of \mathbf{Q}_1 , whose top intervals are contained in some bounded enlargement of $2^k I_{E_2}$. Therefore $\mu_1(F^{-1}(E_2)) \lesssim 2^k \mu_2(E_2)$. Conversely, if E_1 is a generating set in \mathbf{Q}_1 then $F(E_1)$ could be generously covered by $O(2^k)$ generating sets in \mathbf{Q}_2 , and the length of the top intervals of these covering sets are comparable to $|I_{E_1}|$. Therefore $S_{2,lac}(h \circ F)(E_1) \lesssim 2^k S_{2,lac}(h)(F(E_1))$. Therefore by pull-back (essentially the same proof as [6, Proposition 3.2]) we obtain

$$\|h \circ F\|_{\mathcal{L}^{p_2}(\mathbf{Q}_1, S_{2,lac}, \mu_1)} \lesssim 2^{2k} \|h\|_{\mathcal{L}^{p_2}(\mathbf{Q}_2, S_{2,lac}, \mu_2)} .$$

On the other hand, notice that for any $P \in \mathbf{Q}_1$ we have

$$c(P, P') \lesssim_N 2^{-Nk} \frac{1}{|I_P|} \int (\tilde{\chi}_{I_P} \tilde{\chi}_{I_{P'}})^N \sup_{j: N_j \in \omega_{1,P,upper}} |d_j| |f_3|$$

Therefore using (perhaps a version of) Lemma 6.6, it follows that

$$\begin{aligned} \|c\|_{\mathcal{L}^{p_3}(\mathbf{Q}_1, S_{\infty}, \mu_1)} &\lesssim_N 2^{-Nk} \sup_{R \in \overline{\mathbf{Q}_1}} \left(\frac{1}{|I_R|} \int_{F_3} (\tilde{\chi}_{I_R} \tilde{\chi}_{I_{R'}})^{p'_3 N} \right)^{1/p'_3} \|f_3\|_{p_3} \\ &\lesssim_N 2^{-Nk} M_N(F_3) \|f_3\|_{p_3} , \end{aligned}$$

here we used $p_3 > r/(r-2)$. This leads to the desired estimate for $N_{diag,k}$.

Consequently, for the purpose of proving the desired estimates for N the roles of \mathbf{Q}_1 and \mathbf{Q}_2 are symmetric, and we may assume that

$$M_N(\mathbf{Q}_1, F_3) \mu_1(\mathbf{Q}_1) \leq M_N(\mathbf{Q}_2, F_3) \mu_2(\mathbf{Q}_2) .$$

It follows from Lemma 8.5 that

$$\begin{aligned} &\|Kh\|_{\mathcal{L}^{p'_1}(\mathbf{Q}_1, S_{2,lac}, \mu_1)} \\ &\lesssim (M_N(\mathbf{Q}_1, F_1)^{1/p_1+1/p_2} \mu_1(\mathbf{Q}_1)^{1/p_2} + M_N(F_3) \mu_2(\mathbf{Q}_2)^{1/p_2}) \|h\|_{\mathcal{L}^\infty(S_{2,lac})} \|f_3\|_{p_3} \\ &\lesssim M_N(F_3) \mu_2(\mathbf{Q}_2)^{1/p_2} \|h\|_{\mathcal{L}^\infty(S_{2,lac})} \|f_3\|_{p_3} . \end{aligned}$$

Recall that on \mathbf{Q}_1 and \mathbf{Q}_2 we have $S_2 \lesssim S_{2,lac}$, so using a combination of outer Radon-Nikodym and outer Hölder inequalities, we obtain

$$\begin{aligned} N_{offdiag} &= \sum_{P \in \mathbf{Q}_1} |I_P| g(P) (Kh)(P) \\ &\lesssim \|g\|_{\mathcal{L}^{p_1}(\mathbf{Q}_1, S_{2,lac}, \mu_1)} \|Kh\|_{\mathcal{L}^{p'_1}(\mathbf{Q}_1, S_{2,lac}, \mu_1)} \\ &\lesssim M_N(F_3) \mu_1(\mathbf{Q}_1)^{1/p_1} \mu_2(\mathbf{Q}_2)^{1/p_2} \|g\|_{\mathcal{L}^\infty(S_{2,lac})} \|h\|_{\mathcal{L}^\infty(S_{2,lac})} \|f_3\|_{p_3} \\ &\lesssim M_N(F_3) \left(\sum_m |I_{E_m}| \right)^{1/p_1} \left(\sum_n |I_{G_n}| \right)^{1/p_2} \|g\|_{\mathcal{L}^\infty(S_{2,lac})} \|h\|_{\mathcal{L}^\infty(S_{2,lac})} \|f_3\|_{p_3} , \end{aligned}$$

as desired. \square

Lemma 8.5. *Let f be supported on \mathbf{Q}_2 . Then*

$$\begin{aligned} &\|Kf\|_{\mathcal{L}^{p'_1}(\mathbf{Q}_1, S_{2,lac}, \mu_1)} \\ &\lesssim (M_N(\mathbf{Q}_1, F_3)^{1/p'_3} \mu_1(\mathbf{Q}_1)^{1/p_2} + M_N(F_3) \mu_2(\mathbf{Q}_2)^{1/p_2}) \|f\|_{\mathcal{L}^\infty(S_{2,lac})} \|f_3\|_{p_3} . \end{aligned}$$

To prove Lemma 8.5, we first show the following estimate for $\|Kf\|_\infty$.

Lemma 8.6. *Uniform over $\mathbf{P} \subset \mathbf{P}_1$, it holds for any $s < 2$ and $N > 0$ that*

$$\begin{aligned} & \|Kf\|_{\mathcal{L}^\infty(\mathbf{P}, S_{2,lac}, \mu_1)} \\ & \lesssim_{s,N} m_{\frac{r}{r-2}, N}(\mathbf{P}, (f_3 d_j)) \|f\|_{\mathcal{L}^\infty(\mathbf{P}_2, S_{2,lac}, \mu_2)} + m_{s,N}(\mathbf{P}, (f_3 d_j T_j^* f)) \end{aligned}$$

Below we deduce Lemma 8.5 from Lemma 8.6. By interpolation, it suffices to prove weak-type estimates. We may assume $\|f\|_{\mathcal{L}^\infty(\mathbf{Q}_2, S_{2,lac}, \mu_2)} = 1$ by scaling. Using Lemma 6.6 for $s = p_1$, for any $\lambda > 0$ we may find $A \subset \mathbf{Q}_1$ such that

$$\begin{aligned} m_{p'_1, N}(\mathbf{Q}_1 \setminus A, (f_3 d_j T_j^* f)) & \leq \lambda, \text{ and} \\ \lambda \mu_1(A)^{1/p'_1} & \lesssim M_N(\mathbf{Q}_1, 1_{F_3})^{1/p_1} \left\| \left(\sum_j |f_3 d_j T_j^* f|^{p'_1} \right)^{1/p'_1} \right\|_{p'_1}. \end{aligned}$$

Thus, arguing as in Section 8.1.1 we obtain

$$\begin{aligned} \lambda \mu_1(A)^{1/p'_1} & \lesssim M_N(\mathbf{Q}_1, F_3)^{1/p_1} M_N(\mathbf{P}_2, F_3)^{1/p_2} \|f\|_{\mathcal{L}^{p_2}(\mathbf{Q}_2, S_{2,lac}, \mu_2)} \|f_3\|_{p_3} \\ & \lesssim M_N(F_3) \mu_2(\mathbf{Q}_2)^{1/p_2} \|f_3\|_{p_3}. \end{aligned}$$

Since $p_3 > r/(r-2)$, it follows from Lemma 6.6 that we could find $B \subset \mathbf{Q}_1$ with

$$m_{\frac{r}{r-2}}(\mathbf{Q}_1 \setminus B, (f_3 d_j)) \leq \lambda \text{ and}$$

$$\lambda \mu_1(B)^{1/p_3} \lesssim M_N(\mathbf{Q}_1, 1_{F_3})^{1/p'_3} \|f_3\|_{p_3}.$$

Clearly we have $\mu_1(B) \leq \mu_1(\mathbf{Q}_1)$, therefore

$$\lambda \mu_1(B)^{1/p'_1} = \lambda \mu_1(B)^{1/p_3} \mu_1(B)^{1/p_2} \lesssim M_N(\mathbf{Q}_1, 1_{F_3})^{1/p'_3} \|f_3\|_{p_3} \mu_1(\mathbf{Q}_1)^{1/p_2},$$

and this completes our proof of Lemma 8.5.

Proof of Lemma 8.6. Let $E \subset \mathbf{P}$ be lacunary. Using sparseness of \mathbf{P}_2 , it is clear that $\mathbf{P}_2(E)$ is a lacunary subset of \mathbf{P}_2 . Let $u_j := f_3 d_j T_j^* f$ for convenience. Similar to the proof of Lemma 6.7, it suffices to show that if $\|g\|_{\mathcal{L}^\infty(E, S_{2,lac}, \mu_1)} = 1$ then

$$\begin{aligned} & \frac{1}{|I_E|} \sum_{P \in E} |I_P| Kf(P) g(P) \\ & \lesssim_{s,N} m_{\frac{r}{r-2}, N}(\mathbf{P}, (f_3 d_j)) \|f\|_{\mathcal{L}^\infty(\mathbf{P}_2, S_{2,lac}, \mu_2)} + m_{s,N}(\mathbf{P}, (u_j)). \end{aligned}$$

Let \mathbf{J} and b_P be defined as usual, we also start with

$$(32) \quad \sum_{P \in E} |I_P| Kf(P) g(P) \lesssim \sum_J \left\| \sum_{P \in E} |I_P| g(P) \phi_{1,P} b_P \right\|_{L^1(J)} \leq A + B,$$

where A denote the contribution of $|I_P| \leq 2|J|$ and the rest is in B . Using $|b_P(x)| \leq \sup_{j: N_j \in \omega_{1,P,upper}} |u_j|$, it is not hard to see that

$$A \lesssim |I_E| m_{\infty, N}(E, (u_j)).$$

We now estimate B . For convenience, let $G_E(x) = \sum_{P \in E} |I_P| g(P) \phi_{1,P}(x)$ and

$$V^s(g_E)(x) := \sup_{K, n_0 < \dots < n_K} \left(\sum_{j=1}^K |(\Pi_{n_j} - \Pi_{n_{j-1}}) g_E(x)|^s \right)^{1/s}$$

for $0 < s < \infty$, where Π_j denotes a suitable Fourier projection on to the relevant frequency scale of E (see also proof of Lemma 6.7).

For each J , let $\omega_J = \bigcup_{P \in E: |I_P| \geq 4|J|} \omega_{1,P,upper}$. As in the proof of Lemma 6.7 there exists $R \in \bar{\mathbf{P}}$ with $|I_R| \approx |J|$, $\text{dist}(I_R, J) \lesssim |J|$, and $\omega_J \subset \tilde{\omega}_R$. By decomposing $T_{j,|I_P|/16} = T_{j,|J|/4} + (T_{j,|I_P|/16} - T_{j,|J|/4})$ we obtain the decomposition $b_P = \ell_{P,J} + s_{P,J}$, and we will estimate the contribution of each term.

Contribution of $\ell_{P,J}$: We decompose further $\ell_{P,J} = \ell_{P,J,core} + \ell_{P,J,tail}$ by decomposing $f = f_{I_E,core} + f_{I_E,tail}$ where $f_{I_E,core}$ is the restriction of f to the subset of $\mathbf{P}_2(E)$ containing all P' with $I_{P'} \subset 5I_E$. By the Hölder inequality, we have

$$\sum_J \left\| \sum_{|I_P| \geq 4|J|} |I_P| g(P) \phi_{1,P} \ell_{P,J,core} \right\|_{L^1(J)} \lesssim \sum_J \|B_J(g, f_{I_E,core})\|_{L^1(J)} \left(\sum_j |f_3 d_j|^{\frac{r}{r-2}} \right)^{\frac{r-2}{r}} ,$$

$$B_J(g, f) := \left(\sum_j \left| \sum_{P, P' \text{ constraints}} |I_P| |I_{P'}| g(P) f(P') \tilde{\phi}_{1,P,j} \tilde{\phi}_{2,P',j} \right|^{r/2} \right)^{2/r} ,$$

the constraints in the sum are $|I_{P'}| \leq |I_P|/16$, $|I_P| \geq 4|J|$, $|I_{P'}| \geq |J|/4$. Let $F_{E,core} = \sum_{P'} |I_{P'}| f_{I_E,core}(P') \phi_{2,P'}$ and define $F_{E,tail}$ similarly. Clearly,

$$B_J(g, f_{I_E,core})(x) \lesssim M_J(V^{r/2}(G_E, F_{E,core})) , \quad (r > 2 \text{ is needed for convexity}) ,$$

where as in [5] we define the bilinear variation-norm $V^{r/2}(h_1, h_2)$ to be

$$\sup_{K: N_0 < \dots < N_K} \left(\sum_{k=1}^K \left| \sum_{N_{k-1} < i < j \leq N_k} (\Delta_i h_1)(\tilde{\Delta}_j h_2) \right|^{r/2} \right)^{2/r} ,$$

and $(\Delta_j)_{j \geq 0}$ and $(\tilde{\Delta}_j)_{j \geq 0}$ are two suitable families of Littlewood-Paley projections relative to ξ_E . By variation-norm estimates for paraproducts [5], it follows that

$$\begin{aligned} & \sum_J \left\| \sum_{|I_P| \geq 4|J|} |I_P| \phi_{1,P} \ell_{P,J,core} \right\|_{L^1(J)} \\ & \lesssim m_{\frac{r}{r-2}, N}(E, (f_3 d_j)) \|M(V^{r/2}(G_E, F_{E,core}))\|_{L^{1,\infty}(3I_E)} \\ (33) \quad & \lesssim m_{\frac{r}{r-2}, N}(E, (f_3 d_j)) \|G_E\|_{L^{p'}(\mathbb{R})} \|F_{E,core}\|_{L^p(\mathbb{R})} . \end{aligned}$$

By Lemma 6.4 we have $\|F_{E,core}\|_{L^p(\mathbb{R})} \lesssim |I_E|^{1/p} \|f\|_{L^\infty(\mathbf{P}_2, S_{2,lac}, \mu_2)}$ and a similar estimate for $\|G_E\|_{L^{p'}(\mathbb{R})}$, giving the desired estimate for the contribution of $\ell_{P,J,core}$.

For contribution of $\ell_{P,J,tail}$, notice that for every $x \in J \subset 3I_E$ it holds that

$$\begin{aligned} B_J(g, f_{E,tail})(x) & \lesssim \left(\sum_j \left| \sum_P |I_P| g(P) \tilde{\phi}_{1,P,j} \right|^r \right)^{1/r} \left(\sum_{P'} |I_{P'}| |f_{I_E,tail}(P') \phi_{2,P'}| \right) \\ & \lesssim_N M_J(V^r(G_E)) \sup_{P': I_{P'} \cap 4I_E = \emptyset, |I_{P'}| \leq |I_E|/16} |f(P')| \tilde{\chi}_{I_{P'}}(c(I_E))^{N+5} . \end{aligned}$$

It follows that

$$\begin{aligned} & \sum_J \left\| \sum_{P \in E: |I_P| \geq 4|J|} |I_P| g(P) \phi_{1,P} \ell_{P,J,tail} \right\|_{L^1(J)} \lesssim \\ (34) \quad & \lesssim m_{\frac{r}{r-2}, N}(E, (f_3 d_j)) \|M(V^r(G_E))\|_{L^{1,\infty}(3I_E)} \sup_{P'} |f(P')| \tilde{\chi}_{I_{P'}}(c(I_E))^{N+5} , \end{aligned}$$

which implies the desired estimate for the contribution of $\ell_{P,J,tail}$ via the continuous Lépingle inequality (see e.g. [1, 9]).

Contribution of $s_{P,J}$: Since $|I_P| \geq 4|J| \geq 16|I_{P'}|$, we could remove the constraint $|I_{P'}| \leq |I_P|/16$ in $s_{P,J}$. Using the Hölder inequality, it follows that

$$\begin{aligned} & \sum_J \left\| \sum_{P \in E: |I_P| \geq 4|J|} |I_P| g(P) \phi_{1,P} s_{P,J} \right\|_{L^1(J)} \lesssim \\ & \lesssim \sum_J \left\| \left(\sum_j \left| \sum_{|I_P| \geq 4|J|} |I_P| g(P) \tilde{\phi}_{1,P,j}^{s'} \right|^{s'} \right)^{1/s'} \left(\sum_{j: N_j \in \omega_J} |u_j|^s \right)^{1/s} \right\|_{L^1(J)} \\ & \lesssim \sum_J |J| M_J(V^{s'}(G_E)) m_{s,N}(E, (u_j)) \quad . \end{aligned}$$

The desired estimate then follows from the continuous Lépingle inequality. \square

8.2. Proof of Theorem 8.1.

8.2.1. The basic range. We first show the range $2 < q_1, q_2 < 2r/(4-r)$ and $1 < q_2 < r/2$ of Theorem 8.1. Note that now $2r/(3r-4) < q'_1 < 2$ and $q'_3 > r/(r-2)$. Using outer Radon Nikodym/Hölder inequalities and Lemma 8.2, it follows that

$$\begin{aligned} \langle T_{CC}(f_1, f_2), f_3 \rangle &= \Lambda_{\mathbf{P}_1, \mathbf{P}_2} := \sum_{P \in \mathbf{P}_1} |I_P| a_1(P) K(a_2)(P) \\ (35) \qquad \qquad \qquad &\lesssim M_N(F_3) \|a_1\|_{\mathcal{L}^{q_1}} \|a_2\|_{\mathcal{L}^{q_2}} \|f_3\|_{\mathcal{L}^{q'_3}(\mathbb{R})} \end{aligned}$$

where in the above display (and in subsequent displays) the outer norm for a_j is over $(\mathbf{P}_j, S_{2,lac}, \mu_j)$ and the outer norm for Ka_2 is over $(\mathbf{P}_1, S_{1,overlap} + S_{2,lac}, \mu_1)$. Thus the desired claim follows from generalized Carleson embeddings and duality, and the trivial bound $M_N(F_3) \lesssim 1$.

8.2.2. Extending the range. Let $\mathbf{M} \subset \{\alpha_1 + \alpha_2 + \alpha_3 = 1\}$ have the following vertices

$$\begin{aligned} & M_1\left(\frac{3r-4}{2r}, \frac{1}{2}, -\frac{r-2}{r}\right) \quad , \quad M_2\left(\frac{1}{2}, \frac{3r-4}{2r}, -\frac{r-2}{r}\right) \\ & M_3\left(0, \frac{3r-4}{2r}, \frac{4-r}{2r}\right) \quad , \quad M_4(0, 0, 1) \quad , \quad M_5\left(\frac{3r-4}{2r}, 0, \frac{4-r}{2r}\right) \quad . \end{aligned}$$

Similar to Section 7.3.2, to show Theorem 8.1 it suffices to prove restricted-weak type estimates for one $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in any neighborhood of any given vertex of \mathbf{M} . Our starting point will be (35), where \mathbf{P}_1 and \mathbf{P}_2 are symmetric, thus it suffices to consider M_1, M_3, M_4 . Below let F_1, F_2, F_3 have finite positive Lebesgue measures. Let $C_0 > 0$ be a finite large absolute constant such that $\|M(f)\|_{1,\infty} < \frac{C_0}{4} \|f\|_1$ in the maximal inequality.

In the following, let $2 < p_1, p_2 < 2r/(4-r)$ and $p_3 > \frac{r}{r-2}$ with $\sum 1/p_j = 1$.

Near M_1 . Let $B = \bigcup_{j=1,2} \{M(1_{F_j}) > C_0 |F_j|/|F_3|\}$, clearly $|B| < |F_3|/2$. Let $G_3 = F_3 \setminus B$. We may assume that for some $k_1, k_2 \geq 0$ it holds that

$$(36) \qquad 2^{k_1} \leq 1 + \frac{\text{dist}(I_P, B^c)}{|I_P|} < 2^{k_1+1} |I_P| \quad , \quad \forall P \in \mathbf{P}_1$$

and similarly $2^{k_2} \leq 1 + \frac{\text{dist}(I_{P'}, B^c)}{|I_{P'}|} < 2^{k_2+1}$ for every $P' \in \mathbf{P}_2$, provided that we have sufficient decay in the estimate. Let $|f_1| \leq 1_{F_1}$, $|f_2| \leq 1_{F_2}$, and $|f_3| \leq 1_{F_3-B}$.

By convexity, it follows from (35) and Carleson embeddings that

$$\begin{aligned}\Lambda_{\mathbf{P}_1, \mathbf{P}_2} &\lesssim |F_3|^{\frac{1}{p_3}} \prod_{j=1,2} M_N(\mathbf{P}_j, 1_{F_3 \setminus B})^{\frac{1}{p_j}} \|a_j\|_{\infty}^{1-\frac{2}{p_j}} \|a_j\|_{2,\infty}^{\frac{2}{p_j}} \\ &\lesssim 2^{-N(k_1+k_2)} |F_1|^{1-\frac{1}{p_1}} |F_2|^{1-\frac{1}{p_2}} |F_3|^{\frac{1}{p_3}-(1-\frac{2}{p_1})-(1-\frac{2}{p_2})}\end{aligned}$$

thus $\Lambda_{\mathbf{P}_1, \mathbf{P}_2}$ satisfies restricted weak-type estimate for $\alpha = (1/p'_1, 1/p'_2, -1/p_3)$, which could be made arbitrarily close to M_1 .

Near M_3 . Let $B = \{M(1_{F_3}) > C_0|F_3|/|F_1|\}$, clearly $|B| < |F_1|/2$. We may assume that (36) holds for some $k_1 \geq 0$, provided that we have sufficient decay in the estimate. Let $|f_1| \leq 1_{F_1-B}$ and $|f_2| \leq 1_{F_2}$ and $|f_3| \leq 1_{F_3}$, we will show that

$$(37) \quad \Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2, f_3) \lesssim |F_2|^{1-\frac{1}{p_2}} |F_3|^{\frac{1}{p_2}}$$

which implies desired estimates by letting p_2 close to $2r/(4-r)$.

To show (37), we first use convexity and (35) and Carleson embeddings to obtain

$$\begin{aligned}\Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2, f_3) &\lesssim |F_3|^{\frac{1}{p_3}} M_N(\mathbf{P}_1, 1_{F_3})^{\frac{1}{p_1}} \|a_1\|_{\infty}^{1-\frac{2}{p_1}} \|a_1\|_{2,\infty}^{\frac{2}{p_1}} M_N(\mathbf{P}_2, F_3)^{1/p_2} \|a_2\|_{p_2} \\ &\lesssim 2^{-Nk_1} |F_3|^{\frac{1}{p_3}} \left(\frac{|F_3|}{|F_1|}\right)^{\frac{1}{p_1}} |F_1|^{\frac{1}{p_1}} M_N(\mathbf{P}_2, F_3)^{\frac{1}{p_2}} \|a_2\|_{p_2} \\ (38) \quad &\lesssim 2^{-Nk_1} M_N(\mathbf{P}_2, F_3)^{1/p_2} \|a_2\|_{p_2} |F_3|^{\frac{1}{p_2}}.\end{aligned}$$

It follows in particular that $\Lambda(f_1, f_2, f_3) \leq 2^{-Nk_1} |F_2|^{1/p_2} |F_3|^{1/p'_2}$ for any $2 < p_2 < 2r/(4-r)$, which would imply the desired estimate (37) if $|F_3| \leq |F_2|$. When $|F_3| > |F_2|$ we will carry out essentially another layer of restricted weak-type interpolation, which we details below. Let $\tilde{B}_1 = F_3 \cap \{M1_{F_2} > C_0|F_2|/|F_3|\}$, clearly $|\tilde{B}_1| < |F_3|/2$. We will show that

$$(39) \quad \Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2, f_3 1_{\tilde{B}_1^c}) \lesssim 2^{-Nk_1} |F_2|^{1-1/p_2} |F_3|^{1/p_3}.$$

To show (39), we may assume that for some $k_2 \geq 0$ it holds for every $P' \in \mathbf{P}_2$ that $2^{k_2} \leq 1 + \frac{\text{dist}(I_{P'}, \tilde{B}_1^c)}{|I_{P'}|} < 2^{k_2+1}$, provided that we have enough decay in the estimates. It follows from (38) that

$$\begin{aligned}\Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2, f_3 1_{\tilde{B}_1^c}) &\lesssim 2^{-Nk_1} M_N(\mathbf{P}_2, F_3 - \tilde{B}_1)^{\frac{1}{p_2}} \|a_2\|_{\infty}^{1-\frac{2}{p_2}} \|a_2\|_{2,\infty}^{\frac{2}{p_2}} |F_3|^{\frac{1}{p_2}} \\ &\lesssim 2^{-Nk_1} 2^{-Nk_2} \left(\frac{|F_2|}{|F_3|}\right)^{1-\frac{2}{p_2}} |F_2|^{\frac{1}{p_2}} |F_3|^{\frac{1}{p_2}} \\ &\lesssim 2^{-N(k_1+k_2)} |F_2|^{1-\frac{1}{p_2}} |F_3|^{\frac{1}{p_2}}\end{aligned}$$

proving (39). Now, if $|\tilde{B}_1| > |F_2|$ then we continue to let $\tilde{B}_2 = \tilde{B}_1 \cap \{M1_{F_2} > C_0|F_2|/|\tilde{B}_1|\}$ and similarly $\tilde{B}_3, \dots, \tilde{B}_m$ such that $|F_3| > 2|\tilde{B}_1| > \dots > 2^m|\tilde{B}_m|$ until $|\tilde{B}_m| \leq |F_2|$. (This process has to stop since $|F_3|/|F_2|$ is finite.) We obtain (here

for convenience of notation let $\tilde{B}_0 \equiv F_3$):

$$\begin{aligned} \Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2, f_3) &\lesssim |\Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2, f_3 1_{\tilde{B}_m})| + \sum_{j=0}^{m-1} |\Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2, f_3 1_{\tilde{B}_j - \tilde{B}_{j+1}})| \\ &\lesssim 2^{-Nk_1} |F_2|^{\frac{1}{p_2}} |\tilde{B}_m|^{1-\frac{1}{p_2}} + \sum_{j=0}^{m-1} 2^{-Nk_1} |F_2|^{1-\frac{1}{p_2}} (2^{-j} |F_3|)^{\frac{1}{p_2}} \\ &\lesssim 2^{-Nk_1} |F_2|^{1-\frac{1}{p_2}} |F_3|^{\frac{1}{p_2}}, \quad \text{as desired.} \end{aligned}$$

Near M_4 . Let $B = \{M1_{F_3} > C_0|F_3|/|F_1|\}$ which satisfies $|B| < |F_1|/3$. Let $|f_1| \leq 1_{F_1-B}$, $|f_2| \leq 1_{F_2}$, $|f_3| \leq 1_{F_3}$, it suffices to show that

$$(40) \quad \Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2, f_3) \lesssim_\epsilon |F_2|^\epsilon |F_3|^{1-\epsilon},$$

for any $0 < \epsilon \leq 1/p_2$. As before, we will assume (36). For any $2 < p_2 < 2r/(4-r)$ we also have (38), which would imply the desired estimate if $|F_3| \geq |F_2|$. When $|F_3| < |F_2|$ we will use interpolation. Similar to the analysis near M_3 , it suffices to show that if $\tilde{B}_1 = F_2 \cap \{M1_{F_3} > C_0|F_3|/|F_2|\}$ (which satisfies $|\tilde{B}_1| < |F_2|/2$) then

$$\Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2 1_{\tilde{B}_1^c}, f_3) \lesssim 2^{-Nk_1} |F_2|^\epsilon |F_3|^{1-\epsilon}$$

and we will again assume that for some $k_2 \geq 0$ it holds for every $P' \in \mathbf{P}_2$ that $2^{k_2} \leq 1 + \frac{\text{dist}(I_{P'}, \tilde{B}_1^c)}{|I_{P'}|} < 2^{k_2}$. Now it follows from (38) that

$$\begin{aligned} \Lambda_{\mathbf{P}_1, \mathbf{P}_2}(f_1, f_2 1_{\tilde{B}_1^c}, f_3) &\lesssim 2^{-Nk_1} M_N(\mathbf{P}_2, F_3)^{\frac{1}{p_2}} \|a_2\|_\infty^{1-\frac{2}{p_2}} \|a_2\|_{2,\infty}^{\frac{2}{p_2}} |F_3|^{\frac{1}{p_2}} \\ &\lesssim 2^{-Nk_1} 2^{-Nk_2} \min(1, \frac{|F_3|}{|F_2|})^{\frac{1}{p_2}} |F_2|^{\frac{1}{p_2}} |F_3|^{\frac{1}{p_2}} \\ &\lesssim 2^{-N(k_1+k_2)} |F_2|^\epsilon |F_3|^{1-\epsilon}. \end{aligned}$$

This completes the proof of Theorem 8.1. \square

9. ESTIMATES FOR LM MODEL OPERATORS

Theorem 9.1. *Let $T(f_1, f_2)$ be an LM model operator. Then $\|T\|_{L^{q_1} \times L^{q_2} \rightarrow L^{q_3}} < \infty$ for all $1/q_3 = 1/q_1 + 1/q_2$ with $q_1, q_2 > 2r/(3r-4)$ and $q_3 > r'/2$.*

Proof. We sketch the proof of this theorem, which is a simple bilinear extension of the proof of Theorem 6.9. For a tritile $Q \in \mathbf{Q}$ in the definition of T we let ω_Q be the convex hull of $\omega_{Q_1}, \omega_{Q_2}, \omega_{Q_3}$. Let $D^{-1}x = x/2$ the dilation by $1/2$ with respect to the origin, and define $\omega_{Q, \text{lower}} = \omega_{1,Q, \text{lower}} \cap \omega_{2,Q, \text{lower}} \cap D^{-1}(\omega_{3,Q, \text{lower}})$ and define $\omega_{Q, \text{upper}}$ similarly. Without loss of generality we may assume that $\omega_{Q, \text{lower}}$ is a half-line and $\omega_{Q, \text{upper}}$ is a finite interval for every $Q \in \mathbf{Q}$. We now could define $\tilde{\omega}_Q$ to be the convex hull of $20\omega_Q$ and $20\omega_{Q, \text{upper}}$, and from here we may define generating subsets of \mathbf{Q} , and construct outer measure spaces on \mathbf{Q} using the usual outer measure and sizes as in Section 6.1.

Let A denote the collection of intervals J such that for some $P_1, P_2 \in \mathbf{Q}$ we have $I_{P_1} \subset J \subset 30I_{P_2}$. Let $p_1, p_2 > 2$ and $p_3 > r/(r-2)$ such that $\sum 1/p_j = 1$; note

that this implies $p_1, p_2 < 2r/(4-r)$. By routine applications of outer measure techniques and embedding theorems in Section 6, we obtain

$$\langle T(f_1, f_2), f_3 \rangle \lesssim_N |F_3|^{\frac{1}{p_3}} \sup_{J \in A} M_J(1_{\text{supp}(f_3)} \tilde{\chi}_J^N)^{1-\frac{1}{p_3}} \prod_{j=1,2} |F_j|^{\frac{1}{p_j}} \sup_{J \in A} M_J(f_j \tilde{\chi}_J^N)^{1-\frac{2}{p_j}}.$$

Using similar arguments as in previous sections, it follows that restricted weak-type estimates holds for at least one $(\alpha_1, \alpha_2, \alpha_3)$ in any neighborhood of any of the following points, which then implies the theorem (below $\beta_r := (3r-4)/(2r)$):

$$H_1(\beta_r, -\beta_r, 1) \ , \ H_2(-\beta_r, \beta_r, 1) \ , \ H_3(1/2, \beta_r, 1/2-\beta_r) \ , \ H_4(\beta_r, 1/2, 1/2-\beta_r) \ . \ \square$$

APPENDIX A. REDUCTION TO DISCRETE OPERATORS

We will show below that the (variation-norm) bilinear Fourier operators with symbols m_{CC} , $m_{C \times C}$, m_{BC} , m_{CC} are controlled by the respective discrete operators.

Recall that A is the set of all admissible triple $(side, m, n)$ in \mathcal{H} . It is clear that

- If $\alpha \in A$ then $L_1 \leq \min(m_\alpha, n_\alpha) \leq 2L_1$.
- If $k > 2L_1$ then $(left, 2L_1, k)$ and $(right, k, 2L_1)$ are not in A .

Let \mathbf{D}_{left} , \mathbf{D}_{right} be the sets of left and right-sided dyadic intervals. By definition, if $\alpha = (side, m, n)$ then $\mathbf{I}_{M,N}(\alpha)$ consists of $I \in \mathbf{D}_{side}$ with $M \in I_{lower,\alpha} := I - (m+1)|I|$ and $N \in I_{upper,\alpha} := I + (n+1)|I|$.

A.1. Discretization of m_{CC} . We will discuss the discretization for $m_{CC,1}$, the discretization for other $m_{CC,j}$ are similar. We first make several observations regarding the decomposition of $1_{M < \xi < N}$ in Section 3.1.1.

Lemma A.1 (Observation 1). *If $I \in \mathbf{I}(side, m, n)$ and $\xi \in \frac{5}{4}I$ then*

$$(41) \quad m/(4n) < |\xi - M|/|\xi - N| < 4m/n \ .$$

Proof. This follows from $(m-1/8)|I| \leq |\xi - M|, |\xi - N| \leq (m+2+1/8)|I|$. \square

Lemma A.2 (Observation 2). *Let $I \in \mathbf{I}(side, m, n)$ with $n \geq 4m$. Assume that $J \in \mathbf{I}(side', m', n')$ and $\sup J < \inf I$. Then $n'/m' > n/m$.*

Similarly, if $m \geq 4n$ and J is on the right of I then $m'/n' > m/n$.

Proof. Assume $n \geq 4m$, the other case is symmetric. Without loss of generality we may assume that J is adjacent to I . Since $n > m$, it follows that $|J| = |I|$ or $|J| = |I|/2$. In the first case the desired estimate is trivial, and in the second case we have $m' \leq 2m$ and $n' \geq 2n+2$, which also implies the desired estimate. \square

As a corollary of Lemma A.2, it follows that the intervals in A_2 are strictly in between the elements of A_1 and the elements of A_3 .

The next observation concerns cancellation when summing bump functions ϕ_α over the set of $\alpha \in A$ such that $I \in \mathbf{I}_{M,N}(\alpha)$, where M, N, I are fixed.

Lemma A.3 (Observation 3). (i) Given any $C \geq 4$ we can find $O(L_1)$ many tuples $(a_j, b_j, u_j, v_j) \in \mathbb{Z}^2 \times \{1/2, 1, 2\}^2$, with $L_1 + 1 \leq a_j, b_j \leq L_1$, such that the following holds for every left dyadic interval I and every $M < N$:

$$\sum_{\alpha \in A: m_\alpha \geq Cn_\alpha} \phi_\alpha(\xi) 1_{I \in \mathbf{I}_{M,N}(\alpha)} = \sum_j \phi_{u_j, v_j}(\xi) 1_{M < c(I) - (a_j - 1/2)|I|} 1_{N \in I + b_j|I|} \quad .$$

Furthermore, an analogous statement also holds for right dyadic intervals.

(ii) A similar statement also holds for $\sum_{m_\alpha \leq n_\alpha/C}$.

Remark: the key idea here is that the left hand sides in the above equalities involve infinitely many terms, while the right hand sides contain only $O(L_1)$ terms.

Proof. Let I_- and I_+ be the left and right neighbors of I in \mathcal{H} . Below we only consider the $\sum_{m_\alpha \geq Cn_\alpha}$, the other sum could be handled similarly.

Since $C \geq 4$ we have $m_\alpha \geq 4L_1$, while $2L_1 \geq n_\alpha \geq L_1$. Our key observation is the fact that: in the sum, the ratios $u(I) = |I_-|/|I|$ and $v(I) = |I_+|/|I|$ depends only on $side_\alpha$ and n_α . In other words, knowing the *side* of I and knowing n_α (only a finite number of possible values) we could determine $\phi_\alpha(\xi) \equiv \phi_{u(I), v(I)}(\xi)$ completely and thus we have the freedom to sum the indicator constraints on M , $1_{M \in I - (m+1)|I|}$, over $m \geq Cn_\alpha$. The following table details this observation

side of I	values of n_α	Corresponding values of $(u(I), v(I))$
left	L_1	$(2, \frac{1}{2})$
left	$L_1 < n_\alpha \leq 2L_1$	$(2, 1)$
right	L_1	$(1, \frac{1}{2})$
right	$L_1 < n_\alpha \leq 2L_1$	$(2, 1)$

(note that if I is right-sided then $n_\alpha < 2L_1$ by definition of A). \square

A.1.1. *Discretization for $m_{CC,1}$:* Using Lemma A.3, we may decompose $m_{CC,1}$ into

$$\sum_{|I| \leq |J|/16} \phi_I(\xi_1) \varphi_J(\xi_2) 1_{M < c(I) - a_1|I|, N \in I + b_1|I|} 1_{M < c(J) - a_2|J|, N \in J + b_2|J|}$$

in the sum I and J belong to fixed collection of dyadic intervals, ϕ_I, φ_J are given bump functions supported in $(5/4)I$ and $(5/4)J$, and $L_1 \leq a_1, b_1, a_2, b_2 \leq L_1$.

It follows from a standard Fourier sampling argument that we can decompose

$$\iint e^{ix(\xi_1 + \xi_2)} \phi_I(\xi_1) \varphi_J(\xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) d\xi_1 d\xi_2$$

into finitely many wavelet sums

$$\sum_{P, P' \text{ tiles: } \omega_P = I, \omega_{P'} = J} |I_P| |I_{P'}| \langle f_1, \phi_{1,P} \rangle \langle f_2, \phi_{2,P'} \rangle \phi_{1,P}(x) \phi_{2,P'}(x) \quad ,$$

where $\phi_{1,P}, \phi_{2,P'}$ are L^1 -normalized wave functions adapted to the tiles P, P' . Thus the resulting bilinear operator for $m_{CC,1}$ is controlled by a finite sum over CC discrete operators.

A.2. Discretization of m_{BC} . For convenience let $\xi_3 = -\xi_1 - \xi_2$ below. Thanks to fast decay of a_k , it suffices to consider the contribution of one fixed k . In other words, it suffices to consider symbols

$$\tilde{m}_{BC} = \sum_{\alpha \in A} \sum_{\ell(S) \leq |I|} \phi_{1,S}(\xi_1) \phi_{2,S}(\xi_2) \phi_{3,S}(-\xi_3) \phi_{\alpha,I}(-\xi_3) 1_{\{2M \in I_{lower,\alpha}\}} 1_{\{2N \in I_{upper,\alpha}\}}$$

in the sum S is in a fixed collection of shifted dyadic cubes with the property (10), and I is in a fixed collection of dyadic intervals, and $\phi_{j,S}$ are uniformly C^m bump functions supported in $\frac{5}{6}S_i$, where n could be chosen arbitrarily large.

For any $I, S, \alpha \in A$, and any Schwarz f_1, f_2, f_3 , we have

$$\begin{aligned} & \int \int \int e^{ix(\xi_1 + \xi_2)} \phi_{3,S}(\xi_1 + \xi_2) \phi_{\alpha,I}(\xi_1 + \xi_2) \left(\prod_{j=1}^2 \phi_{j,S}(\xi_j) \widehat{f_j}(\xi_j) d\xi_1 d\xi_2 \right) f_3(x) dx \\ &= \int \int \int_{\xi_1 + \xi_2 + \xi_3 = 0} \phi_{\alpha,I}(-\xi_3) \phi_{3,S}(-\xi_3) \widehat{f_3}(\xi_3) \left(\prod_{j=1}^2 \widehat{f_j}(\xi_j) \phi_{j,S}(\xi_j) \right) d\xi_1 d\xi_2 d\xi_3 \\ &= C \int \left(f_3 * \widehat{\phi_{3,S}} * \widehat{\phi_{\alpha,I}} \right) (2y) \prod_{j=1}^2 \left(f_j * \widetilde{\phi_{j,S}} \right) (y) dy \\ &= C \sum_{m \in \mathbb{Z}} \int_0^1 \frac{1}{\ell(S)} \left(f_3 * \widehat{\phi_{3,S}} * \widehat{\phi_{\alpha,I}} \right) \left(\frac{2(t-m)}{\ell(S)} \right) \prod_{j=1}^2 \left(f_j * \widetilde{\phi_{j,S}} \right) \left(\frac{t-m}{\ell(S)} \right) dt \\ &= C \int_0^1 \sum_{Q \in \mathbf{Q}_{t,S}} |I_Q| \left\langle f_3 * \widehat{\phi_{\alpha,I}}, \widehat{\phi_{3,Q}} \right\rangle \prod_{j=1}^2 \langle f_j, \phi_{j,Q} \rangle dt, \quad \text{where} \end{aligned}$$

- for each $t \in [0, 1)$, $\mathbf{Q}_{t,S}$ is the collection of all tritiles $Q = (Q_1, Q_2, Q_3)$ with

$$Q_1 = I_Q \times S_1, \quad Q_2 = I_Q \times S_2, \quad Q_3 = I_Q \times S_3,$$

and I_Q is a shifted dyadic interval of length $\ell(S)^{-1}$ (with t -dependent shift);

- for any $Q \in \mathbf{Q}_{t,S}$ and for each $j = 1, 2, 3$, define $\phi_{j,Q}(x) := \widetilde{\phi_{j,S}}(x - c(I_Q))$ which are L^1 -normalized wave packets adapted to $Q_j = I_Q \times S_j$ (with frequency support in $\frac{5}{6}S_j$).

Recall that for every $S \in \mathbf{S}$ the distance between S_1 and S_2 is comparable to $L_2|\ell(S)| \sim L_1|\ell(S)|$ (due to the Whitney condition (10)). Clearly, the collection \mathbf{Q}_t will be of rank 1 (with uniform constants over $0 \leq t \leq 1$) if L_1 is sufficiently large.

Now, for any b_3 -shifted dyadic interval I , by a Fourier sampling argument ($f_3 * \widehat{\psi_{\alpha,I}}(x)$ equals a sum over finitely many terms of the form $\sum_{P \in \mathbf{P}_I} \langle f_3, \overline{\phi_P} \rangle \overline{\phi_P}(x)$, where \mathbf{P}_I is the collection of all rectangles $P = J \times I$ formed using standard dyadic intervals J of length $|I|^{-1}$, and $\{\phi_P, P \in \mathbf{P}_I\}$ is a collection of Fourier wave packets adapted to $P \in \mathbf{P}_I$ with $\text{supp}(\widehat{\phi_{J \times I}}) \subset \frac{5}{4}I$).

It follows that, modulo a multiple by some absolute constant,

$$\int \left(\int \int e^{ix(\xi_1 + \xi_2)} \tilde{m}_{BC} \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) d\xi_1 d\xi_2 \right) f_3(x) dx$$

can be decomposed into finitely many terms of the following form:

$$\int_0^1 \sum_{Q \in \mathbf{Q}_t} |I_Q| \langle f_1, \phi_{1,Q} \rangle \langle f_2, \phi_{2,Q} \rangle \langle \mathcal{C}(f_3), \overline{\phi_{3,Q}} \rangle dt \quad , \quad \text{where}$$

$$\mathcal{C}(f_3)(x) := \sum_{\alpha, I} \sum_{P \in \mathbf{P}_I} 1_{\{2M \in I_{lower, \alpha}\}} 1_{\{2N \in I_{upper, \alpha}\}} \langle f_3, \overline{\phi_P} \rangle \overline{\phi_P(x)} \quad ,$$

and $\mathbf{Q}_t := \bigcup_{S \in \mathbf{S}} \mathbf{Q}_{t,S}$. Let \mathbf{P} be the union of \mathbf{P}_I over $I \in \mathbf{D}$. Using Lemma A.3, we can write $\mathcal{C}(f_3)(x)$ as a sum over finitely many terms of the form

$$\pm \sum_{P \in \mathbf{P}} |I_P| \langle f_3, \overline{\phi_P} \rangle \overline{\phi_P(x)} 1_{\{2M \in \omega_{P, lower}\}} 1_{\{2N \in \omega_{P, upper}\}}$$

where $\{(\omega_{P, lower}, \omega_P, \omega_{P, upper}), P \in \mathbf{P}\}$ is rigid. Thus to bound the resulting bilinear operator for m_{BC} , it suffices to estimate discrete model BC operators.

A.3. Discretization of m_{LM} . In Lemma A.4 and Lemma A.5 we will show that the supports of m_{CC} and m_{BC} are inside $\{M \leq \xi_1 \leq \xi_2 \leq N\}$ and they do not intersect except at possibly $(\xi_1, \xi_2) = (M, M)$ and (N, N) . It will follow that

$$(42) \quad m_{LM} = (\chi_{M < \xi_1 < \xi_2 < N} - m_{CC})(\chi_{M < \xi_1 < \xi_2 < N} - m_{BC}) \quad ,$$

which will be used in subsection A.3.3 to reduce the bilinear multiplier operator with symbol m_{LM} to LM model operators.

A.3.1. Support of m_{CC} . It will follow from Lemma A.4 below that $m_{CC} = 1$ on R_1 (defined in (7)), and m_{CC} is supported inside the following enlargement of R_1 :

$$R'_1 := \left\{ (\xi_1, \xi_2) \in [M, N]^2 : \min(|\xi_1 - M|, |\xi_2 - N|) \leq \frac{1}{3}|\xi_1 - \xi_2| \right\} \quad .$$

Note that $\overline{R_1} \subset R'_1$, and $R'_1 \subset \{M \leq \xi_1 < \xi_2 \leq N\}$.

Lemma A.4. (i) If a summand in (8) is non-zero for some $(\xi_1, \xi_2) \in R_1$, then this summand must appear in one of $m_{CC,k}$, $1 \leq k \leq 5$.

(ii) All summands of $m_{CC,k}$ are supported in R'_1 .

Proof. (i) Suppose that for some α, β and $I \in \mathbf{I}(\alpha)$ and $J \in \mathbf{I}(\beta)$ and $(\xi_1, \xi_2) \in R_1$ we have $\phi_{\alpha, I}(\xi_1)\phi_{\beta, J}(\xi_2) \neq 0$.

Without loss of generality we may assume that $(\alpha, \beta) \notin A_1 \times A_3$, so $\alpha \notin A_1$ or $\beta \notin A_3$. By symmetry, we may assume that $\beta \notin A_3$.

We first show that $\alpha \in A_1$. Since $\beta \notin A_3$, it follows from Lemma A.1 that $|\xi_2 - M| \leq 16|\xi_2 - N|$, therefore

$$|\xi_2 - N| \geq |M - N|/15 \geq |\xi_1 - \xi_2|/15 \quad .$$

Since $(\xi_1, \xi_2) \in R_1$, it follows from the definition of R_1 that

$$(43) \quad |\xi_1 - M| \leq |\xi_1 - \xi_2|/200 \leq |M - N|/200 \quad .$$

It follows that $|\xi_1 - M| \leq |\xi_1 - N|/199$. Using Lemma A.1 again, it follows that

$$m_\alpha/n_\alpha \leq 4|\xi_1 - M|/|\xi_1 - N| < 1/4$$

thus $\alpha \in A_1$ as claimed above.

It remains to show that $|I| < |J|/16$. As a corollary of the condition $\alpha \in A_1$ and $\beta \notin A_3$, we have $m_\beta \leq 4n_\beta \leq 8L_1$, while clearly $m_\alpha \geq L_1$. Using $\xi_1 \in \frac{5}{4}I$ and $\xi_j \in \frac{5}{4}J$ and (43) it follows that

$$\frac{|I|}{|J|} \leq \frac{|\xi_1 - M|/(m_\alpha - 1/8)}{|\xi_2 - M|/(m_\beta + 9/8)} \leq \frac{11|\xi_1 - M|}{|\xi_2 - M|} < \frac{1}{16}.$$

This completes the proof of claim (i).

(ii) Without loss of generality we may assume that $\alpha \in A_1$.

By the definition of $m_{CC,j}$'s we have $|I| \leq |J|/16$. It suffices to show that

$$(5/4)I \times (5/4)J \subset R'_1.$$

To see this, take any $(\xi_1, \xi_2) \in \frac{5}{4}I \times \frac{5}{4}J$.

Since $\alpha \in A_1$ implies that $m_\alpha = \min(m_\alpha, n_\alpha) \leq 2L_1$, while clearly $m_\beta \geq L_1 \geq 1$. It follows that

$$\frac{|\xi_1 - M|}{|\xi_2 - M|} \leq \frac{|I|(m_\alpha + 9/8)}{|J|(m_\beta - 1/8)} \leq \frac{1}{4}.$$

Thus $|\xi_1 - M| \leq |\xi_1 - \xi_2|/3$, as desired. \square

A.3.2. Support of m_{BC} . Consider the following enlargement of R_2 (defined in (9)):

$$R'_2 := \left\{ (\xi_1, \xi_2) : |\xi_1 - \xi_2| \leq \frac{1}{2} \min \left(\left| \frac{1}{2}(\xi_1 + \xi_2) - M \right|, \left| \frac{1}{2}(\xi_1 + \xi_2) - N \right| \right) \right\}$$

Lemma A.5. *Let m_{BC} be defined by Definition 3.2. Then:*

- (i) m_{BC} is supported inside R'_2 .
- (ii) Any summand of (12) whose support intersects R_2 must appear in m_{BC} .

Proof. (i) Take any (ξ_1, ξ_2) in the support of m_{BC} , then for some $S \in \mathbf{S}$ and $I \in \mathbf{I}_{2M, 2N}(\alpha)$, $\alpha \in A$, such that $\ell(S) \leq |I|$ we have

$$\phi_{1,S,k}(\xi_1)\phi_{2,S,k}(\xi_2)\phi_{3,S,k}(\xi_1 + \xi_2)\phi_{\alpha,I}(\xi_1 + \xi_2) \neq 0.$$

It follows that $(\xi_1, \xi_2) \in \frac{5}{6}S_1 \times \frac{5}{6}S_2$, therefore using (10) we obtain

$$|\xi_1 - \xi_2| = \sqrt{2} \text{dist}((\xi_1, \xi_2), \{\xi_1 = \xi_2\}) < 5L_2\ell(S).$$

On the other hand, since $\xi_1 + \xi_2 \in \frac{5}{4}I$ and $m_\alpha, n_\alpha \geq L_1$, we have

$$\min \left(\left| \frac{1}{2}(\xi_1 + \xi_2) - M \right|, \left| \frac{1}{2}(\xi_1 + \xi_2) - N \right| \right) \geq \frac{1}{2}(L_1 - 1/8)|I|.$$

Since $\ell(S) \leq |I|$ and since $L_2 = L_1/40$, it is not hard to check that (ξ_1, ξ_2) satisfies the defining property of R'_2 .

(ii) Suppose that $(\xi_1, \xi_2) \in R_2$ such that

$$\phi_{1,S,k}(\xi_1)\phi_{2,S,k}(\xi_2)\phi_{3,Q,k}(\xi_1 + \xi_2)\phi_{\alpha,I}(\xi_1 + \xi_2) \neq 0.$$

We will show that $\ell(S) \leq |I|$.

First, using $(\xi_1, \xi_2) \in R_2$ and using $\xi_1 + \xi_2 \in (5/4)I$, it follows that

$$\begin{aligned} |\xi_1 - \xi_2| &\leq \frac{1}{100} \min \left(\left| \frac{1}{2}(\xi_1 + \xi_2) - M \right|, \left| \frac{1}{2}(\xi_1 + \xi_2) - N \right| \right) \\ &\leq \frac{1}{100} (2L_1 + 1/8)|I|. \end{aligned}$$

Since $(\xi_1, \xi_2) \in (4/5)S_1 \times (4/5)S_2$, it follows that

$$|\xi_1 - \xi_2| = \sqrt{2} \text{dist}((\xi_1, \xi_2), \{\xi_1 = \xi_2\}) \geq \sqrt{2} L_2 \ell(S) .$$

Collectin estimates and using $L_2 = L_1/40$, it is clear that $\ell(S) \leq |I|$, thus the corresponding summand is part of m_{BC} . \square

A.3.3. Reduction to discrete T_{LM} , part I: decomposition into simpler trilinear symbols. Recall that D_t is the dilation $D_t A := \{2^t x : x \in A\}$.

Lemma A.6. m_{LM} can be decomposed into finitely many symbols of the form

$$\begin{aligned} \sum_{\alpha, \beta, \gamma} \sum_{I \in \mathcal{D}_1} \sum_{J \in \mathcal{D}_2} \sum_{L \in \mathcal{D}_3} 1_{\{\text{constraints on } I, J, L\}} \phi_{1,I}(\xi_1) \phi_{2,J}(\xi_2) \phi_{3,L}(-\xi_1 - \xi_2) \times \\ \times 1_{\{M \in I_{\text{lower}, \alpha}, N \in I_{\text{upper}, \alpha}\}} \times 1_{\{M \in J_{\text{lower}, \alpha}, N \in J_{\text{upper}, \alpha}\}} \times 1_{\{2M \in L_{\text{lower}, \alpha}, 2N \in L_{\text{upper}, \alpha}\}} , \end{aligned}$$

- $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are three fixed collections of dyadic intervals;
- $\phi_{1,I}, \phi_{2,J}, \phi_{3,L}$ are C^n -bump functions adapted to these intervals, with support in $\frac{5}{4}I, \frac{5}{4}J, \frac{5}{4}L$; and n could be chosen arbitrarily large
- the constraints on I, J, L read as follows: the length I, J, L are comparable, and the distance between $I, J, D_{-1}L$ are $O(|I|)$.

Note that the dependence on M and N are only in the last three factors of the summands. To prove Lemma A.6, we analyze the factors in the factorization (42).

Part I: First factor. Using Lemma A.4, it follows that

$$\begin{aligned} \chi_{M < \xi_1 < \xi_2 < N} - m_{CC} &= \chi_{\xi_1 < \xi_2} (\chi_{M < \xi_1, \xi_2 < N} - m_{CC}) \\ (44) \quad &= \chi_{\xi_1 < \xi_2} \sum_{\alpha, \beta \in A} \sum_{(I, J) \in \mathbf{I}(\alpha) \times \mathbf{I}(\beta), \text{ constraints}} \phi_{\alpha, I}(\xi_1) \phi_{\beta, J}(\xi_2) , \end{aligned}$$

the constraints on $(I, J) \in \mathbf{I}(\alpha) \times \mathbf{I}(\beta)$ read as follows:

- If $\alpha \notin A_1$ and $\beta \notin A_3$ then there are no constraints.
- If $\alpha \in A_1$ and $\beta \in A_3$ then no I, J are allowed.
- If $\alpha \in A_1$ and $\beta \notin A_3$ then one requires $|I| > |J|/16$.
- If $\alpha \notin A_1$ and $\beta \in A_3$ then one requires $|J| > |I|/16$.

Recall that \mathcal{H} is the union of $\mathbf{I}(\alpha)$ over $\alpha \in A$. We will show that

Lemma A.7. Assume that $(I, J) \in \mathbf{I}(\alpha) \times \mathbf{I}(\beta)$ contributes to the right hand side of (44). Then (i) $|I| \sim |J|$ and (ii) if $\xi_1 < \xi_2$ and $\phi_{\alpha, I}(\xi_1) \phi_{\beta, J}(\xi_2) \neq 0$ then

$$\text{dist}((\xi_1, \xi_2), \{(M, M), (N, N)\}) \sim L_1 |I| .$$

Proof. (i) Without loss of generality we may assume that the ratio of the lengths of I and J is not in $\{1, \frac{1}{2}, 2\}$. As the lengths of adjacent intervals in \mathbf{I} differ by a ratio of 1 or 2 or $1/2$, it follows that $I \neq J$ and they are not adjacent.

Now, we show that $\sup I \leq \inf J$. Assume towards a contradiction that $\sup J \leq \inf I$. Since there is at least one interval in \mathcal{H} between I and J and since the lengths of adjacent intervals in \mathcal{H} differ by a factor in $\{1/2, 1, 2\}$, it follows that $\inf \frac{5}{4}I > \sup \frac{5}{4}J$. Consequently, $(5/4)I \times (5/4)J \cap \{(\xi_1, \xi_2) : \xi_1 < \xi_2\} = \emptyset$, contradicting the fact that (I, J) contributes to (44).

Now, if $\alpha, \beta \in A_2$ then clearly $|I|$ and $|J|$ are comparable to $|M - N|/L_1$, so they are comparable. Therefore we may assume that either $\alpha \notin A_2$ or $\beta \notin A_2$.

We will show that

$$\text{either } \alpha \in A_1 \text{ or } \beta \in A_3 .$$

Note that from the constraints we must have $(\alpha, \beta) \notin A_1 \times A_3$ (otherwise there won't be any (I, J)), thus the above two properties can not hold simultaneously. Assume towards a contradiction that $\alpha \notin A_1$ and $\beta \notin A_3$. Since $\alpha \notin A_2$ or $\beta \notin A_2$, it follows that $\alpha \in A_3$ or $\beta \in A_1$.

- If $\alpha \in A_3$ then using Lemma A.2 and $\sup I \leq \inf J$ it follows that

$$m_\beta/n_\beta > m_\alpha/n_\alpha \geq 4 ,$$

thus $\beta \in A_3$, contradict to the above assumption.

- If $\beta \in A_1$ then $n_\alpha/m_\alpha \geq n_\beta/m_\beta \geq 4$ and thus $\alpha \in A_1$, contradiction again.

Below, without loss of generality assume that $\alpha \in A_1$. Thus $\beta \in A_1$ or A_2 .

If $\beta \in A_1$ then since J is on the right of I we easily have $|I| \leq |J|$, while $|J| < |I|/16$ by the above constraints. Thus $|I| \sim |J|$.

If $\beta \in A_2$, then since $\bigcup_{\gamma \in A_2} \mathbf{I}(\gamma)$ has $O(1)$ elements of comparable lengths, $|J|$ is comparable to $|I_0|$ the length of the right most interval inside $\bigcup_{\gamma \in A_1} \mathbf{I}(\gamma)$. Note that either $I = I_0$ or I is on the left of I_0 , thus $|I| \leq |J_0|$. Combining with the constraint $|J| < 16|I|$, we obtain $|J| \sim |I|$.

(ii) Let $\alpha, \beta \in A$ such that $I \in \mathbf{I}(\alpha)$ and $J \in \mathbf{I}(\beta)$. It is clear that

$$\begin{aligned} & \text{dist}((\xi_1, \xi_2), \{(M, M), (N, N)\}) \\ & \sim \min(|\xi_1 - M| + |\xi_2 - M|, |\xi_1 - N| + |\xi_2 - N|) . \end{aligned}$$

If $(\alpha, \beta) \in (A_1 \cup A_2)^2$ then using $|I| \sim |J|$ we obtain

$$\begin{aligned} & \min(|\xi_1 - M| + |\xi_2 - M|, |\xi_1 - N| + |\xi_2 - N|) \\ & \sim |\xi_1 - M| + |\xi_2 - M| \sim L_1 |I| . \end{aligned}$$

Similarly, if $(\alpha, \beta) \in (A_2 \cup A_3)^2$ then the desired claim follows. Since $(\alpha, \beta) \notin A_1 \times A_3$ (by the constraints), the remaining case is $(\alpha, \beta) \in A_3 \times A_1$, however this can't happen either since I is on the left of J . \square

Part II: The second factor. For convenience of notation, let $\xi_3 := -\xi_1 - \xi_2$. Thanks to Lemma A.5, we may write

$$\begin{aligned} \chi_{M < \xi_1 < \xi_2 < N} - m_{BC} &= \chi_{M < \xi_1, \xi_2 < N} (\chi_{\xi_1 < \xi_2} \chi_{2M < \xi_1 + \xi_2 < 2N} - m_{BC}) \\ (45) \quad &= \chi_{M < \xi_1, \xi_2 < N} \sum_{k, \alpha} a_k \sum_{S \in \mathbf{S}} \sum_{\substack{\ell(S) \geq 2|L| \\ L \in \mathbf{I}_{-2N, -2M}(\alpha)}} \phi_{\alpha, L}(-\xi_3) \phi_{3, S, k}(-\xi_3) \prod_{j=1}^3 \phi_{j, S, k}(\xi_j) . \end{aligned}$$

Lemma A.8. *Suppose that $(S, L) \in (\mathbf{S}, I_{2M, 2N}(\alpha))$ and contributes to the right hand side of (45). Then (i) $\ell(S) \sim |L|$ and (ii) if $(\xi_1, \xi_2) \in (M, N)^2$ is in the support of the corresponding summand then*

$$\text{dist}((\xi_1, \xi_2), \{(M, M), (N, N)\}) \sim L_1 \ell(S) .$$

Proof. (i) Note that $|L| \leq \ell(S)/2$ by definition, thus it remains to show $\ell(S) = O(|L|)$. Without loss of generality, assume that $|L - 2N| < |L - 2M|$. Then

$$\text{dist}(L, 2N) \sim L_1|L| \quad .$$

Since $\phi_{j,S,k}$ is supported inside $(5/6)S_j$, it follows that $(\frac{5}{6}S_1 \times \frac{5}{6}S_2) \cap (M, N)^2 \neq \emptyset$. Take any (ξ_1, ξ_2) in this intersection. Then $\xi_1 < \xi_2 < N$, therefore using $(\xi_1 + \xi_2) \in \text{supp}(\phi_{\alpha,L}) \subset \frac{5}{4}L$ it follows that

$$\begin{aligned} |\xi_1 - \xi_2| &\leq |\xi_1 - N| + |\xi_2 - N| = |\xi_1 + \xi_2 - 2N| \\ &\leq 3|L| + 2\text{dist}(L, 2N) \lesssim L_1|L| \quad . \end{aligned}$$

Now, let ℓ be the line $\{\xi_1 = \xi_2\}$. Since $(\xi_1, \xi_2) \in S_1 \times S_2$, it follows from (10) that

$$\ell(S) \leq L_2^{-1} \text{dist}((\xi_1, \xi_2), \ell) \lesssim L_2^{-1} |\xi_1 - \xi_2| \lesssim (L_1/L_2) |L| \quad .$$

(Recall that $L_1 = O(L_2)$.) This completes the proof of (i).

(ii) Let A be the distance from (ξ_1, ξ_2) to the set of two corners (M, M) and (N, N) . Then the desired lower bound for A follows from

$$A \gtrsim \text{dist}((\xi_1, \xi_2), \ell) \gtrsim L_1 \ell(S) \quad .$$

Using the triangle inequality and the Whitney property (10), we also have

$$\begin{aligned} A &\leq \text{dist}((\xi_1, \xi_2), \ell) + \text{dist}((\xi_1 + \xi_2)/2, \{M, N\}) \\ &\lesssim L_2 \ell(S) + L_1 |L| \quad . \end{aligned}$$

Thanks to (i) we obtain the upper bound $A \lesssim L_1 \ell(S)$, as desired. \square

Part III (final step). We now examine m_{LM} . Using (44) and Lemma A.7 and Lemma A.8, it follows that $m_{LM}(M, N, \xi_1, \xi_2)$ could be written as

$$\begin{aligned} &= \chi_{\xi_1 < \xi_2} \chi_{M < \xi_1, \xi_2 < N} \left(\sum_{\alpha, \beta} \sum_{I, J} \phi_{\alpha, I}(\xi_1) \phi_{\beta, J}(\xi_2) 1_{\text{constraints on } I, J} \right) \times \\ &\times \left(\sum_{k, \gamma} a_k \sum_{S, L} \phi_{1, S, k}(\xi_1) \phi_{2, S, k}(\xi_2) \phi_{3, S, k}(\xi_1 + \xi_2) \phi_{\gamma, L}(\xi_1 + \xi_2) 1_{\text{constraints on } S, L} \right) \end{aligned}$$

here there are constraints on I, J, S, L relative to α, β, γ . Observe that the summation over (k, γ, S, L) is zero outside $\{\xi_1 < \xi_2\}$, while the summation over (α, β, I, J) is zero outside $\{M < \xi_1, \xi_2 < N\}$. Therefore we could drop the factor $\chi_{\xi_1 < \xi_2} \chi_{M < \xi_1, \xi_2 < N}$ in the right hand side and obtain

$$\begin{aligned} m_{LM} &= \sum_{\alpha, \beta, \gamma, k} a_k \sum_{I \in \mathbf{I}_{M, N}(\alpha)} \sum_{J \in \mathbf{I}_{M, N}(\beta)} \sum_{L \in \mathbf{I}_{2M, 2N}(\gamma)} \sum_{S \in \mathbf{S}} 1_{\text{constraints on } I, J, S, L} \times \\ &\times \left[\phi_{\alpha, I}(\xi_1) \phi_{1, S, k}(\xi_1) \right] \times \left[\phi_{\beta, J}(\xi_2) \phi_{2, S, k}(\xi_2) \right] \times \left[\phi_{\gamma, L}(\xi_1 + \xi_2) \phi_{3, S, k}(\xi_1 + \xi_2) \right] \quad . \end{aligned}$$

Since a_k decays rapidly, for the purpose of proving Lemma A.6 we may drop the summation over k and consider only the contribution of one k .

Recall that D_y denotes the dilation by 2^y with respect to 0, i.e. $D_y \xi = 2^y \xi$. In particular $D_{-1} L = \{\frac{1}{2}x : x \in L\}$. We claim that in any non-zero summand in m_{LM} , it holds that

$$(i) \quad |I| \sim |J| \sim \ell(S) \sim |L|.$$

(ii) The spatial distances between any two of $I, J, S_1, S_2, D_{-1}(S_3), D_{-1}(L)$ are bounded above by $O(L_1|I|)$.

Indeed, (i) follows from Lemma A.7 and Lemma A.8:

$$\begin{aligned}\ell(S) &\sim |L| \sim \frac{1}{L_1} \text{dist}((\xi_1, \xi_2), \{(M, M), (N, N)\}) \\ |I| &\sim |J| \sim \frac{1}{L_1} \text{dist}((\xi_1, \xi_2), \{(M, M), (N, N)\}) \quad .\end{aligned}$$

For (ii), by the Whitney property (10) $\text{dist}(S_1, S_2) = O(L_2\ell(S))$, thus the distances between $D_{-1}(S_3)$ and S_1, S_2 are $O(L_1\ell(S))$. The desired claim now follows from examining the factors in the summation.

It follows that by decomposing m_{LM} into $O(1)$ sums we may assume that J, L, S_1, S_2, S_3 are completely determined from I : comparable length and nearby location.

Since a_k decays rapidly we can ignore the summation over k , and below we will even drop the dependence on k of the inner sums for brevity of notations. We end up with a symbol of the form

$$\sum_{I \in \mathbf{D}_1} \sum_{J \in \mathbf{D}_2} \sum_{L \in \mathbf{D}_3} \psi_{1,I}(\xi_1) \psi_{2,J}(\xi_2) \psi_{3,L}(\xi_1 + \xi_2) 1_{\text{constraints}}$$

here $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$ could be \mathbf{D}_{left} or $\mathbf{D}_{\text{right}}$. are three fixed collections of (standard) dyadic intervals, $\psi_{1,I}$ is a C^n bump function adapted to I and is supported on $\frac{5}{4}I$, and $\psi_{2,J}$ and $\psi_{3,L}$ satisfy similar properties. The constraints read as follows: for fixed bounded integers m_1, n_1, m_2, n_2 it holds that

- $|J| = 2^{m_1}|I|$ and $c(I) + n_1|I| \in J \neq \emptyset$.
- $|L| = 2^{m_2}|I|$ and $c(I) + n_2|I| \in D_{-1}(L)$.
- $M \in I_{\text{lower},\alpha} \cap J_{\text{lower},\beta}$ and $2M \in L_{\text{lower},\gamma}$;
- $N \in I_{\text{upper},\alpha} \cap J_{\text{upper},\beta}$ and $2N \in L_{\text{upper},\gamma}$.

This completes the proof of Lemma A.6. \square

A.3.4. Reduction to T_{LM} , part II: completion of the proof. Using Lemma A.6, we may decompose m_{LM} into boundedly many m_{m_1, n_1, m_2, n_2} , defined by

$$\begin{aligned}&\sum_{\alpha, \beta, \gamma} \sum_{I, J, L} \psi_{1,I}(\xi_1) \psi_{2,J}(\xi_2) \psi_{3,L}(\xi_1 + \xi_2) 1_{\text{constraints on } I, J, L} \times \\&\times 1_{\{M \in I_{\text{lower},\alpha}, N \in I_{\text{upper},\alpha}\}} 1_{\{M \in J_{\text{lower},\beta}, N \in J_{\text{upper},\beta}\}} 1_{\{2M \in L_{\text{lower},\gamma}, 2N \in L_{\text{upper},\gamma}\}} \quad ,\end{aligned}$$

and the ‘constraints’ on I, J, L specify the location and the length of J and L relative to I using m_1, n_1, m_2, n_2 as discussed in the last section. Note that the decomposition of m_{LM} is independent of M and N .

For each fixed I, J, L , we will sum the summands over α, β, γ . Using Lemma A.3, we will divide m_{m_1, n_1, m_2, n_2} into 8 symbols, such that in the summation each of I, J, L are required to be in one of $\mathbf{D}_{\text{left}}, \mathbf{D}_{\text{right}}$, and each of these 8 symbols could

be further decomposed into $O(L_1)$ symbols having the following structure:

$$\sum_{I \in \mathbf{D}_1} \sum_{J \in \mathbf{D}_2} \sum_{L \in \mathbf{D}_3} \psi_{1,I}(\xi_1) \psi_{2,J}(\xi_2) \psi_{3,L}(-\xi_1 - \xi_2) \times \\ \times \mathbf{1}_{\{M \in I_{1,lower}, N \in I_{1,upper}\}} \mathbf{1}_{\{M \in J_{2,lower}, N \in J_{2,upper}\}} \mathbf{1}_{\{2M \in L_{3,lower}, 2N \in L_{3,upper}\}}$$

where ψ_j are bump functions supported in $[-c, c]$, ($1/2 < c < 5/8$) and the collection of interval triples $\{(I_{1,lower}, I, I_{1,upper}), I \in \mathbf{D}_1\}$ is rigid, and the same holds for the other two collections of interval triples. Note that the rigidity type of these collections are not opposite, i.e. we won't have both *(infinite, finite)* and *(finite, infinite)*. Due to the flexibility of *(finite, finite)* rigidity (which could be converted into two terms of any of the other types), we could assume that the rigidity type of the three collections are the same.

We now split the intervals I, J, L if necessary to ensure $|I| = |J| = |L|$: to do this we will split $\psi_{1,I}, \psi_{2,J}, \psi_{3,L}$ and we will correspondingly split all bounded intervals in $I_{1,lower}, I_{1,upper}, J_{2,lower}, J_{2,upper}, L_{3,lower}, L_{3,upper}$ (but we won't split the halflines). Note that if we split $\psi_{1,I}$ into 2^k bump function adapted to the corresponding subintervals of I ($k \geq 0$), then each new bump function in theory could be supported in an interval as large as the $(1 + 2^k(c - 1/2))$ enlargement of the adapting subinterval. Since k is bounded, by choosing c sufficiently close to $1/2$ when partitioning $\mathbf{1}_{M < \xi < N}$ we could ensure that $1 + 2^k(c - 1/2) < 5/4$.

After the splitting, the rest of the discretization is similar to the discretization of m_{BC} . We omit the details.

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